A Landau fluid model for dispersive magnetohydrodynamics

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Abstract

A mono-fluid model with Landau damping is presented for strongly magnetized electron-proton collisionless plasmas whose distribution functions are close to bi-Maxwellians. This description that includes dynamical equations for the gyrotropic components of the pressure and heat flux tensors, extends the Landau-fluid model of Snyder, Hammett and Dorland [Phys. Plasmas **4**, 3974 (1997)] by retaining Hall effect and finite Larmor radius corrections. It accurately reproduces the weakly nonlinear dynamics of dispersive Alfvén waves whose wave lengths are large compared to the ion inertial length, whatever their direction of propagation, and also the rapid Landau dissipation of long magnetosonic waves in a warm plasma.

I. INTRODUCTION

Both in natural and fusion plasmas, collisions are generally negligible, making the usual magnetohydrodynamics questionable. On the other hand, in most situations, direct numerical integrations of the Vlasov-Maxwell equations in three space dimensions are beyond the capabilities of the present day computers when a broad range of scales is involved. The gyrokinetic description[1,2] that averages over the gyrotropic motion of the particles and that is extensively used for fusion plasmas, reduces the number of independent variables but still needs an enormous computational strength.

Situations involving a broad range of scales require a formalism that preserves most of the aspects of a fluid description but includes realistic approximations of the pressure and heat flux tensors. The effect of wave-particle resonances that provide the dominant dissipation processes should in particular be retained. In a collisionless plasma, a fluid behavior can only result from collective constraints, such as the presence of a strong magnetic field. In this case, Chew, Goldberger, and Low[3] first proposed the "double adiabatic laws" or CGL equations for the parallel and perpendicular gyrotropic pressure components, where all the heat fluxes are neglected. The conditions of validity of this assumption are rather stringent.[4] The onset of the mirror instability is for example not correctly described[5] within this approximation that requires a phase velocity much larger than the thermal velocity of the particles. Closures that reproduce linear results from kinetic theory were also proposed but they depend on the equilibrium state and are often presented in Fourier space, leading to the definition of effective polytropic indices.[6,7] In the context of fusion plasmas, an extensive literature was devoted during the last decades to the gyrofluid description[8,9] based on the evolution of hydrodynamic moments obtained from the gyrokinetic equations, and thus also written in a local coordinate system. A hybrid description of low frequency phenomena involving the coupling of a monofluid description with pressure tensors for ions and electrons prescribed by gyrokinetic equations was also developed.[10]

A simplified description more easily amenable to large-scale numerical simulations of a collisionless plasma permeated by a strong magnetic field was suggested by Hammett and coworkers in the form of Landau fluids built to account for wave-particle resonance effects within a MHD framework. The full electromagnetic case is presented by Snyder, Hammett, and Dorland,[11] hereafter referred to as SHD. Hydrodynamic equations for the density and

velocity of the plasma are obtained by taking moments of the microscopic equations. SHD start from guiding center equations but an equivalent derivation can be made from the Vlasov-Maxwell system. The resulting hierarchy must nevertheless be closed and the main work consists in a proper determination of the pressure tensor associated with each species. For the sake of simplicity, an electron-proton plasma is considered in a simple geometry (no curvature drift), with an homogeneous equilibrium state characterized by bi-Maxwellian distribution functions. In its original presentation, the model is limited to scales large enough for both Hall effect and finite Larmor radius (FLR) corrections to be totally negligible. As shown by SHD, this description predicts the correct threshold of the mirror instability. A generalization is however needed in order to consider dispersive MHD turbulence.

Our goal is thus to develop a simple mono-fluid model able to accurately reproduce the weakly nonlinear dynamics of most MHD waves, including kinetic Alfvén waves (KAW) with transverse wavenumber small compared with the inverse proton inertial length. These waves, characterized by an angle of propagation α such that $\cos^2 \alpha \ll \beta$ (where β denotes the squared ratio of the ion acoustic to the Alfvén speeds), are supposed to be produced by the quasi-two-dimensional energy cascade that develops in Alfvén wave turbulence. A simplified model was recently derived[12] and benchmarked by direct comparisons with Vlasov-Maxwell predictions in the limit of long wave-length small-amplitude perturbations. For parallel Alfvén waves, a reductive perturbative expansion of this model reproduces the kinetic derivative nonlinear Schrödinger (KDNLS) equation[13–15] (including its extension to multidimensional wave trains[16]) derived from the Vlasov-Maxwell equations, up to the replacement of the plasma response function by its two- or four-pole Padé approximants. For magnetosonic waves,[17] agreement was obtained with the phase velocity and the Landau damping given in the literature[18] for the regime $\dfrac{m_e}{m_p} \ll \beta \ll \dfrac{T_e}{T_p}$ of adiabatic protons and isothermal electrons with isotropic temperatures. It however turns out that the description of oblique and kinetic Alfvén waves requires a more refined description of the finite Larmor radius effects associated with the non-gyrotropic contributions of both the pressure and heat flux tensors. A first extension[17] of this model was presented in the regime of adiabatic protons and isothermal electrons with small β, that reproduces the classial dispersion and Landau damping of kinetic Alfvén waves in this regime.[18,19] This approach actually involves a heuristic closure relation for the electron pressure that is here recovered as a limiting case of a more general Landau fluid model whose derivation is the main object of the present

paper.

In Section II, the scalings associated with the various MHD waves are explicited and a mono-fluid description of the plasma is obtained, under conditions consistent with the weakly nonlinear regime. In order to describe anisotropic situations, the pressure tensor of each particle species is retained. It includes gyrotropic components that evolve on hydrody-namic time scales, together with non-gyrotropic ones that rapidly adjust to the variations of the hydrodynamic quantities ("slaved" dynamics) and are amenable to a perturbative description (Section III). In Section IV, general closure approximations for the gytrotropic and non gyrotropic heat fluxes are inferred from the kinetic theory of long oblique Alfvén waves presented in Appendices A-C. As mentioned above, this regime that retains the ki-netic effects to leading order, can indeed be viewed as a distinguished limit covering more general situations. The resulting model and its validation are presented in Section V. A few conclusions and projects for further developments are briefly presented in the last section.

II.   AN ASYMPTOTIC FRAMEWORK FOR A FLUID DESCRIPTION

A.   The small amplitude regime

The usual procedure[5] to describe the dynamics of a strongly magnetized collisionless plasma at scales large compared to those of the ion gyro-motion consists in performing an asymptotics (refered to as a $1/\Omega_p$ expansion) where the small parameter is the ratio of the typical considered frequency to the ion gyro-frequency. This approach is appropriate when no smallness assumption is made on the amplitude of the fluctuations, but may be conflicting with the weak-nonlinearity ordering required to close the moment hierarchy. When addressing the weakly nonlinear regime, it is thus preferable to use a unique expansion parameter to characterize the small amplitudes and the long wave lengths and low frequencies of the perturbations. In the distinguished limit that ensures the balance of the nonlinear and dispersive effects, a reductive perturbative expansion then leads to the classical long-wave equations (such as Korteweg-de Vries or derivative nonlinear Schrödinger). This asymptotics may retain terms which are subdominant in an $1/\Omega_p$ expansion that is relative to the scale separation only.

The fluid equations to be derived in this paper are requested to correctly capture the

weakly nonlinear dynamics of dispersive MHD waves, by fitting with the kinetic theory within the ordering prescribed by a reductive perturbative expansion. This approach has the main advantage to separate the various types of waves, retaining only those terms that contribute to their dynamics. It also provides a rigorous framework for a nonlinear theory where some terms are evaluated at the linear level, as for example requested at the level of the heat flux closure.

B. The MHD wave scalings

In a reductive perturbative expansion, the various MHD waves are selected by prescribing different orderings. The ambient magnetic field being taken in the z direction, we assume a propagation in the (x, z) plane along an axis z' making an angle α with the ambient field. For perturbations depending only on z' and propagating at velocity V_0, we define the stretched coordinate $\xi = \epsilon^{1/2}(z' - V_0 t)$.

(i) Oblique magnetosonic waves

The magnetosonic waves are selected by prescribing $b_x = \epsilon b_x^{(1)} + \cdots$, $b_y = \epsilon^{3/2} b_y^{(1)} + \cdots$, $b_z = B_0 + \epsilon b_z^{(1)} + \cdots$, $\rho = \rho^{(0)} + \epsilon \rho^{(1)} + \cdots$, $u_x = \epsilon u_x^{(1)} + \cdots$, $u_y = \epsilon^{3/2} u_y^{(1)} + \cdots$, $u_z = \epsilon u_z^{(1)} + \cdots$, $p_{\perp r} = p_{\perp r}^{(0)} + \epsilon p_{\perp r}^{(1)} + \cdots$, $p_{\parallel r} = p_{\parallel r}^{(0)} + \epsilon p_{\parallel r}^{(1)} + \cdots$, where as usual b is the magnetic field, ρ and u are the density and velocity of the plasma, $p_{\perp r}$ and $p_{\parallel r}$ are the transverse and parallel pressures of the particles of species r. The dispersion and the nonlinearities then act on a slow time $\tau = \epsilon^{3/2} t$.

(ii) Oblique Alfvén waves

The reductive perturbative expansion now involves the scalings $b_y = \epsilon^{1/2}(b_y^{(1)} + \epsilon b_y^{(2)} \cdots)$, $u_y = \epsilon^{1/2}(u_y^{(1)} + \epsilon u_y^{(2)} + \cdots)$, while the previously defined scalings are retained for the other quantities.

(iii) Parallel Alfvén waves

In the case of a propagation angle $\alpha = 0$, one prescribes $b_x = \epsilon^{1/4} b_x^{(1)} + \cdots$, $b_y = \epsilon^{1/4} b_y^{(1)} + \cdots$, $b_z = B_0 + \epsilon^{1/2} b_z^{(1)} + \cdots$, $\rho = \rho^{(0)} + \epsilon^{1/2} \rho^{(1)} + \cdots$, $u_x = \epsilon^{1/4} u_x^{(1)} + \cdots$, $u_y = \epsilon^{1/4} u_y^{(1)} + \cdots$, $u_z = \epsilon^{1/2} u_z^{(1)} + \cdots$, $p_{\perp r} = p_{\perp r}^{(0)} + \epsilon^{1/2} p_{\perp r}^{(1)} + \cdots$, $p_{\parallel r} = p_{\parallel r}^{(0)} + \epsilon^{1/2} p_{\parallel r}^{(1)} + \cdots$, with a slow time $\tau = \epsilon t$.

Furthermore, for all the waves, the gyrotropic heat fluxes are scaled similarly to the pressures. The magnitude of the non-gyrotropic components will be explicited later on,

when these contributions will be considered (Section IV.A).

The above scalings indicate that nonlinear effects comparable to dispersion occur with an amplitude that is smaller for magnetosonic waves than for Alfvén waves. This reflects the longitudinal character of the former waves for which a relatively strong dispersion is requested to arrest shock formation. In contrast, parallel Alfvén waves can support much larger amplitude since they are incompressible. It follows that a weakly nonlinear theory of magnetosonic waves requires a higher order perturbation theory, as it will be discussed in more details in Section V. This delicate situation can nevertheless be prevented by the presence of Landau damping that, when the β of the plasma is not too small, acts on the shortest time scale, making the nonlinear and dispersive corrections subdominant. The small beta limit that makes the electron inertia relevant is in any case out of the scope of a mono-fluid theory. For larger wave amplitude, the lowest order corrections in the usual $1/\Omega_p$ expansion together with a simple Landau-fluid closure for the gyrotropic pressures[11,12] are sufficient. Furthermore, an expansion valid for oblique Alfvén waves, where both the Hall term and non-gyrotropic heat flux components enter at dominant order, will retain all the relevant terms for parallel waves (with possible additional subdominant corrections) and also for magnetosonic waves in the most usual situations. As a consequence of these observations, the construction of the mono-fluid model will be based on the weakly nonlinear dynamics of oblique Alfvén waves with typical wavelengths large compared to the proton inertial length. This approach involves several steps.

C. From a bi-fluid to a mono-fluid description

Starting from the Vlasov-Maxwell equations (A.1)-(A.4) and writing the equations satisfied by the successive moments of the distribution function for particles of species r, one derives an exact hierarchy of fluid equations for the corresponding density $\rho_r = m_r n_r \int f_r d^3 v$, hydrodynamic velocity $u_r = \dfrac{\int v f_r d^3 v}{\int f_r d^3 v}$, pressure tensor $\mathbf{P}_r = m_r n_r \int (v - u_r) \otimes (v - u_r) f_r d^3 v$ and heat flux tensor $\mathbf{Q}_r = m_r n_r \int (v - u_r) \otimes (v - u_r) \otimes (v - u_r) f_r d^3 v$, in the usual form[5]

$$\partial_t \rho_r + \nabla \cdot (\rho_r u_r) = 0 \tag{1}$$

$$\partial_t u_r + u_r \cdot \nabla u_r + \frac{1}{\rho_r} \nabla \cdot \mathbf{P}_r - \frac{q_r}{m_r}\left(e + \frac{1}{c} u_r \times b\right) = 0 \tag{2}$$

$$\partial_t \mathbf{P}_r + \nabla \cdot (u_r \mathbf{P}_r + \mathbf{Q}_r) + \left[\mathbf{P}_r \cdot \nabla u_r + \frac{q_r}{m_r c} b \times \mathbf{P}_r\right]^{\mathcal{S}} = 0 \tag{3}$$

where the tensor $b \times \mathbf{P}_r$ has elements $(b \times \mathbf{P}_r)_{ij} = \epsilon_{iml} b_m P_{r\,lj}$ and where, for a square matrix $\mathbf{A}$, one defines $\mathbf{A}^{\mathcal{S}} = \mathbf{A} + \mathbf{A}^{\mathrm{tr}}$. One has $(b \times \mathbf{P}_r)^{\mathrm{tr}} = -\mathbf{P}_r \times b$. In order to distinguish between scalar and tensorial pressures, bold letters are used to denote tensors of rank two and higher. Coupled to Maxwell equations, such a multi-fluid description resolves the small spatio-temporal scales associated with Langmuir waves that are unneeded when concentrating on the large-scale dynamics of dispersive MHD waves. A mono-fluid description together with the additional approximation of neglecting electron inertia, allows the filtering of these small scales. One is thus led to consider the plasma velocity $u = \dfrac{1}{\rho} \sum_r \rho_r u_r$ where $\rho = \sum_r \rho_r$ is the plasma density and to define the pressure and heat flux tensors associated with each particle species in terms of the deviations from this barycentric velocity, in the form $\mathbf{p}_r = m_r n_r \int (v - u) \otimes (v - u) f_r d^3 v$ and $\mathbf{q}_r = m_r n_r \int (v - u) \otimes (v - u) \otimes (v - u) f_r d^3 v$. One has

$$\mathbf{P}_r = \mathbf{p}_r - \rho_r (u - u_r) \otimes (u - u_r) \tag{4}$$

and

$$Q_{r\,ijk} = q_{r\,ijk} + p_{r\,ij}(u - u_r)_k + p_{r\,ik}(u - u_r)_j + p_{r\,jk}(u - u_r)_i, \tag{5}$$

where the subscripts ijk refer to components of the corresponding tensors.

Defining $\boldsymbol{\delta}_r = \mathbf{P}_r - \mathbf{p}_r$ and

$$\mathbf{R}_r = \nabla \cdot (u_r \boldsymbol{\delta}_r) + \left[\boldsymbol{\delta}_r \cdot \nabla u_r + (\nabla \cdot \mathbf{P}_r) \otimes (u - u_r) \right]^{\mathcal{S}}, \tag{6}$$

one has in Eq. (3)

$$\nabla \cdot (u_r \mathbf{P}_r + \mathbf{Q}_r) + \left[\mathbf{P}_r \cdot \nabla u_r \right]^{\mathcal{S}} = \nabla \cdot (u \mathbf{p}_r + \mathbf{q}_r) + \left[\mathbf{p}_r \cdot \nabla u \right]^{\mathcal{S}} + \mathbf{R}_r. \tag{7}$$

For the orderings involved in the reductive perturbative analysis of the various MHD waves discussed above, neglecting $\boldsymbol{\delta}_r$ and $\mathbf{R}_r$ contributions in the equation for $\mathbf{p}_r$ is possible if the expansion of this quantity is limited to orders strictly lower than ϵ^2 for oblique Alfvén waves and $\epsilon^{5/2}$ for magnetosonic waves. This leads to replace Eq.(3) by

$$\partial_t \mathbf{p}_r + \nabla \cdot (u \mathbf{p}_r + \mathbf{q}_r) + \left[\mathbf{p}_r \cdot \nabla u + \frac{q_r}{m_r c} b \times \mathbf{p}_r \right]^{\mathcal{S}} = 0. \tag{8}$$

Furthermore, one easily gets that

$$\partial_t \rho + \nabla \cdot (u \rho) = 0 \tag{9}$$

and

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \cdot \mathbf{p} - \frac{1}{c} j \times b = 0, \tag{10}$$

where $\mathbf{p} = \sum_r \mathbf{p}_r$ denotes the total pressure tensor and where the electric current $j = \sum_r q_r n_r \int v f_r d^3 v = \sum_r \frac{q_r}{m_r} \rho_r u_r$ is given by $j = \frac{c}{4\pi} \nabla \times b$. In this derivation, we neglect the displacement current and also make the approximation of quasi-neutrality $\sum_r \frac{q_r \rho_r}{m_r} = 0$, as usual when considering slow motion of fluid elements of size greater than the Debye length.[20]

The current j obeys

$$\partial_t j + \nabla \cdot \left(\sum_r \frac{q_r \rho_r}{m_r} u_r \otimes u_r \right) + \sum_r \frac{q_r}{m_r} \nabla \cdot \mathbf{P}_r - \sum_r \frac{q_r^2 \rho_r}{m_r^2} \left(e + \frac{1}{c} u_r \times b \right) = 0. \tag{11}$$

Using the identity

$$\sum_r \frac{q_r \rho_r}{m_r} u_r \otimes u_r = u \otimes j + j \otimes u - \sum_r \frac{q_r \rho_r}{m_r} u \otimes u + \sum_r \frac{q_r \rho_r}{m_r} (u_r - u) \otimes (u_r - u), \tag{12}$$

and the fact that for a plasma of protons and electrons of electric charge $q_p = -q_e = q$,

$$\sum_r \frac{q_r^2}{m_r^2} \rho_r u_r = q^2 \left(\frac{1}{m_e} + \frac{1}{m_p} \right) \frac{\rho u}{m_e + m_p} - q \left(\frac{m_p}{m_e} - \frac{m_e}{m_p} \right) \frac{j}{m_e + m_p}, \tag{13}$$

one gets[20]

$$\partial_t j + \nabla \cdot \left(u \otimes j + j \otimes u - \sum_r \frac{q_r \rho_r}{m_r} u \otimes u \right) + \sum_r \frac{q_r}{m_r} \nabla \cdot \mathbf{p}_r$$
$$- \sum_r \frac{q_r^2 \rho_r}{m_r^2} e - \frac{q^2}{c} \left(\frac{1}{m_e} + \frac{1}{m_p} \right) \frac{\rho\, u \times b}{m_e + m_p} + \frac{q}{c} \left(\frac{m_p}{m_e} - \frac{m_e}{m_p} \right) \frac{j \times b}{m_e + m_i} = 0. \tag{14}$$

This equation simplifies when terms involving the ratio $\frac{m_e}{m_p}$ are neglected and quasi-neutrality is assumed, which leads to write $\rho_r = m_r n$ and $u \approx u_p$. One obtains

$$\partial_t j + \nabla \cdot (u \otimes j + j \otimes u) - \frac{q^2 n}{m_e} \left(e + \frac{u \times b}{c} - \frac{j \times b}{nqc} + \frac{1}{qn} \nabla \cdot \mathbf{p}_e \right) = 0. \tag{15}$$

For small nonlinearity and under the assumption $\beta \gg \frac{m_e}{m_i}$, the two first terms of the above equation are subdominant. From Maxwell equation (A.2), one then obtains the induction equation

$$\partial_t b - \nabla \times (u \times b) = -\frac{c m_p}{q} \nabla \times \left(\frac{1}{4\pi\rho} (\nabla \times b) \times b - \frac{1}{\rho} \nabla \cdot \mathbf{p}_e \right), \tag{16}$$

that includes the Hall term together with the effect of the electron pressure.

Equations (9), (10), (8) and (16) constitute a closed system, provided a closure approximation is made to express the heat fluxes. Nevertheless, a direct resolution of Eq. (8) would have to resolve time scales associated with the gyro-motion of the particles, a condition that is practically impossible to achieve in numerical simulations that also retain hydrodynamical scales. As shown in Section III, this scale separation can in fact be used to define a reduced description where the evolution of the gyrotropic components of the pressure tensors is followed on hydrodynamic time scales,while the non-gyrotropic ones obey a slaved dynamics in the sense that they are prescribed by the instantaneous values of hydrodynamic quantities. A similar separation can be made at the level of the heat fluxes that contribute to the gyrotropic pressures through both gyrotropic and non-gyrotropic components. Again the gyrotropic heat fluxes require a closure approximation taking the form of dynamical equations, while the non-gyrotropic ones are slaved (Section IV).

III.   THE PRESSURE TENSORS

In order to isolate the gyrotropic components of the pressure tensor, it is convenient to rewrite Eq. (8) for the pressure tensor of each particle species in the form

$$\mathbf{p}_r \times \widehat{b} - \widehat{b} \times \mathbf{p}_r = \mathbf{k}_r \tag{17}$$

where $\widehat{b} = \dfrac{b}{|b|}$ is the unit vector along the local magnetic field and

$$\mathbf{k}_r = \frac{1}{\Omega_r} \frac{B_0}{|b|} \left[\frac{d\mathbf{p}_r}{dt} + (\nabla \cdot u)\mathbf{p}_r + \nabla \cdot \mathbf{q}_r + (\mathbf{p}_r \cdot \nabla u)^{\mathcal{S}} \right]. \tag{18}$$

In this equation, B_0 denotes the amplitude of the ambient field and $\Omega_r = \dfrac{q_r B_0}{m_r c}$ is the gyrofrequency of the particles of species r. Furthermore, $\dfrac{d}{dt} = \partial_t + u \cdot \nabla$ denotes the convective derivative.

A few classical results are first recalled for completeness.[21,22] We first note that the left hand side of Eq. (17) can be viewed as a self-adjoint linear operator acting on $\mathbf{p}_r$, whose kernel is spanned by the tensors $(\mathbf{I} - \widehat{b} \otimes \widehat{b})$ and $\widehat{b} \otimes \widehat{b}$. It is thus convenient to define the projection $\overline{\mathbf{a}}$ of any (3×3) rank two tensor $\mathbf{a}$ on the image of this operator as

$$\overline{\mathbf{a}} = \mathbf{a} - \frac{1}{2}\mathbf{a} : (\mathbf{I} - \widehat{b} \otimes \widehat{b})(\mathbf{I} - \widehat{b} \otimes \widehat{b}) - (\mathbf{a} : \widehat{b} \otimes \widehat{b})\widehat{b} \otimes \widehat{b}, \tag{19}$$

which implies $\operatorname{tr}\overline{\mathbf{a}} = 0$ and $\overline{\mathbf{a}} : \widehat{b} \otimes \widehat{b} = 0$. Here $\mathbf{I}$ is the identity matrix and the double contraction of two square matrices $\mathbf{m}$ and $\mathbf{n}$ is defined as $\mathbf{m} : \mathbf{n} = \sum_{ij} m_{ij} n_{ij}$. In particular, the pressure tensor is written as the sum $\mathbf{p}_r = \mathbf{p}_r^G + \boldsymbol{\pi}_r$ of an element of the kernel

$$\mathbf{p}_r^G = \frac{1}{2}\mathbf{p}_r : (\mathbf{I} - \widehat{b} \otimes \widehat{b})(\mathbf{I} - \widehat{b} \otimes \widehat{b}) + (\mathbf{p}_r : \widehat{b} \otimes \widehat{b})\widehat{b} \otimes \widehat{b} \tag{20}$$

$$\equiv p_{\perp r}(\mathbf{I} - \widehat{b} \otimes \widehat{b}) + p_{\|r}\widehat{b} \otimes \widehat{b} \tag{21}$$

and of a non-gyrotropic component $\boldsymbol{\pi}_r = \overline{\mathbf{p}}_r$ that thus satisfies $\operatorname{tr}\boldsymbol{\pi}_r = 0$ and $\boldsymbol{\pi}_r : \widehat{b} \otimes \widehat{b} = 0$.

A. Dynamics of the gyrotropic pressures

To obtain the equations for the gyrotropic pressure components, one applies the trace and the contraction with $\widehat{b} \otimes \widehat{b}$ on both sides of Eq. (17) to get

$$\operatorname{tr}\frac{d\mathbf{p}_r}{dt} + (\nabla \cdot u)\operatorname{tr}\mathbf{p}_r^G + \operatorname{tr}(\nabla \cdot \mathbf{q}_r) + \operatorname{tr}(\mathbf{p}_r^G \cdot \nabla u)^S + s_{1r} = 0 \tag{22}$$

with $s_{1r} = \operatorname{tr}(\boldsymbol{\pi}_r \cdot \nabla u)^S$ and

$$\frac{d\mathbf{p}_r}{dt} : \widehat{b} \otimes \widehat{b} + \left((\nabla \cdot u)\mathbf{p}_r^G + \nabla \cdot \mathbf{q}_r + (\mathbf{p}_r^G \cdot \nabla u)^S\right) : \widehat{b} \otimes \widehat{b} + s_{2r} = 0 \tag{23}$$

with $s_{2r} = (\boldsymbol{\pi}_r \cdot \nabla u)^S : \widehat{b} \otimes \widehat{b}$.

The trace and the time derivative commute but this is not the case for the contraction with $\widehat{b} \otimes \widehat{b}$. One writes

$$\frac{d\mathbf{p}_r}{dt} : \widehat{b} \otimes \widehat{b} = \frac{d}{dt}(\mathbf{p}_r : \widehat{b} \otimes \widehat{b}) - \mathbf{p}_r : \frac{d}{dt}(\widehat{b} \otimes \widehat{b}) = \frac{dp_{\|r}}{dt} - s_{3r} \tag{24}$$

where, using $\left(\frac{d\widehat{b}}{dt} \otimes \widehat{b} + \widehat{b} \otimes \frac{d\widehat{b}}{dt}\right) : (\mathbf{I} - \widehat{b} \otimes \widehat{b}) = 0$ and $\left(\frac{d\widehat{b}}{dt} \otimes \widehat{b} + \widehat{b} \otimes \frac{d\widehat{b}}{dt}\right) : \widehat{b} \otimes \widehat{b} = 0$, one has $s_{3r} = \boldsymbol{\pi}_r : \frac{d}{dt}(\widehat{b} \otimes \widehat{b})$. One thus gets generalized CGL equations that include heat fluxes and coupling to the non-gyrotropic components of the pressure tensors,

$$\partial_t p_{\perp r} + \nabla \cdot (u\, p_{\perp r}) + p_{\perp r}\nabla \cdot u - p_{\perp r}\widehat{b} \cdot \nabla u \cdot \widehat{b} + \frac{1}{2}\left(\operatorname{tr}\nabla \cdot \mathbf{q}_r - \widehat{b} \cdot (\nabla \cdot \mathbf{q}_r) \cdot \widehat{b}\right)$$

$$+ \frac{1}{2}(s_{1r} - s_{2r} + s_{3r}) = 0 \tag{25}$$

$$\partial_t p_{\|r} + \nabla \cdot (u\, p_{\|r}) + 2p_{\|r}\widehat{b} \cdot \nabla u \cdot \widehat{b} + \widehat{b} \cdot (\nabla \cdot \mathbf{q}_r) \cdot \widehat{b} + s_{2r} - s_{3r} = 0. \tag{26}$$

One easily checks that for the scalings defined in Sections II.A and the non-gyrotropic pressure components given in Section III.B, the couplings s_{1r}, s_{2r} and s_{3r} to the non-gyrotropic

pressure components are negligible. Note that similar equations for the gyrotropic pressures can be obtained in a bi-fluid description, up to the replacement of the plasma velocity u by that of the individual species u_r. It is noticeable that in the present derivation based on the hypothesis of weak nonlinearity together with long spatial and temporal scales, the parallel and transverse pressures decouple from the non-gyrotropic pressure components but are sensitive to the gyrotropic and non-gyrotropic components of the heat fluxes $\mathbf{q}_r$ that can both contribute to the gyrotropic components of $\nabla \cdot \mathbf{q}_r$.

B. Non-gyrotropic pressure contributions

In order to determine the non-gyrotropic contributions to the pressure tensor of the various particle species, we start from Eq. (17). Using the solvability conditions provided by the equations for the gyrotropic pressures, it is rewritten

$$\boldsymbol{\pi}_r \times \widehat{b} - \widehat{b} \times \boldsymbol{\pi}_r = \overline{\mathbf{k}}_r \tag{27}$$

where $\mathbf{k}_r$ can be decomposed into the sum of a contribution involving the gyrotropic pressures and the heat fluxes

$$\boldsymbol{\kappa}_r = \frac{1}{\Omega_r} \frac{B_0}{|b|} \left[\frac{d\mathbf{p}_r^G}{dt} + (\nabla \cdot u)\mathbf{p}_r^G + \nabla \cdot \mathbf{q}_r + (\mathbf{p}_r^G \cdot \nabla u)^S \right] \tag{28}$$

and of a term linear in $\boldsymbol{\pi}_r$

$$L(\boldsymbol{\pi}_r) = \frac{1}{\Omega_r} \frac{B_0}{|b|} \left[\frac{d\boldsymbol{\pi}_r}{dt} + (\nabla \cdot u)\boldsymbol{\pi}_r + (\boldsymbol{\pi}_r \cdot \nabla u)^S \right]. \tag{29}$$

In $\overline{\boldsymbol{\kappa}_r}$, the second term of the r.h.s. of Eq. (28) does not contribute, while the first one rewrites

$$\overline{\frac{d\mathbf{p}_r^G}{dt}} = (p_{\|r} - p_{\perp r})\frac{d}{dt}(\widehat{b} \otimes \widehat{b}) = (p_{\|r} - p_{\perp r})\frac{1}{|b|^2}\left(\frac{db}{dt} \otimes b + b \otimes \frac{db}{dt} - \frac{2}{|b|}\frac{d|b|}{dt} b \otimes b \right) \tag{30}$$

that is explicited by using the induction equation (16).

It is then convenient to split the non-gyrotropic pressure as $\boldsymbol{\pi}_r = \boldsymbol{\pi}_{r,1} + \boldsymbol{\pi}_{r,2}$ with

$$\boldsymbol{\pi}_{r,1} \times \widehat{b} - \widehat{b} \times \boldsymbol{\pi}_{r,1} = \overline{\boldsymbol{\kappa}_r} \tag{31}$$

$$\boldsymbol{\pi}_{r,2} \times \widehat{b} - \widehat{b} \times \boldsymbol{\pi}_{r,2} = \overline{L(\boldsymbol{\pi}_{r,1})} + \overline{L(\boldsymbol{\pi}_{r,2})}. \tag{32}$$

In a weakly nonlinear regime, the quantity $L(\boldsymbol{\pi}_r)$ is of higher order than $\boldsymbol{\pi}_r$, which enables one to neglect $L(\boldsymbol{\pi}_{r,2})$ in Eq. (32). We restrict ourselves to this level of approximation since

pushing further the above perturbative calculation would conflict with the approximations made for the derivation of the pressure equation (8) used in a mono-fluid description. The above equations can be solved in the form[21]

$$\boldsymbol{\pi}_{r,1} = \frac{1}{4}\left[\widehat{b} \times \overline{\boldsymbol{\kappa}_r} \cdot (\mathbf{I} + 3\widehat{b} \otimes \widehat{b})\right]^{\mathcal{S}} \tag{33}$$

$$\boldsymbol{\pi}_{r,2} = \frac{1}{4}\left[\widehat{b} \times \overline{L(\boldsymbol{\pi}_{r,1})} \cdot (\mathbf{I} + 3\widehat{b} \otimes \widehat{b})\right]^{\mathcal{S}} \tag{34}$$

where the overlines turn out not to be necessary in the above formulae. These expressions are nevertheless cumbersome to be used in a numerical code.

In some situations, the contribution $\boldsymbol{\pi}_{r,1}$ is sufficient and can even be simplified by approximating $\widehat{b}$ by the unit vector $\widehat{z}$ along the ambient magnetic field. This leads to define $\boldsymbol{\pi}_r^{[1]}$ by

$$\boldsymbol{\pi}_r^{[1]} \times \widehat{z} - \widehat{z} \times \boldsymbol{\pi}_r^{[1]} = \overline{\overline{\boldsymbol{\chi}_r^{[1]}}} \tag{35}$$

together with $\widehat{z} \cdot \boldsymbol{\pi}_r^{[1]} \cdot \widehat{z} = 0$ and $\boldsymbol{\pi}_r^{[1]} : \mathbf{I} = 0$, where

$$\boldsymbol{\chi}_r^{[1]} = \frac{1}{\Omega_r}\left[\left(\frac{d\mathbf{p}_r^G}{dt}\right)^{[1]} + (\mathbf{p}_r^{G[1]} \cdot \nabla u)^{\mathcal{S}} + \nabla \cdot \mathbf{q}_r\right] \tag{36}$$

with $\mathbf{p}_r^{G[1]} = p_{\perp r}(\mathbf{I} - \widehat{z} \otimes \widehat{z}) + p_{\|r}\widehat{z} \otimes \widehat{z}$ and

$$\left(\frac{d\mathbf{p}_r^G}{dt}\right)^{[1]} = \frac{dp_{\perp r}}{dt}(\mathbf{I} - \widehat{z} \otimes \widehat{z}) + \frac{dp_{\|r}}{dt}\widehat{z} \otimes \widehat{z} + (p_{\|p} - p_{\perp p})\partial_z\left[u \otimes \widehat{z} - (\widehat{z} \cdot u)\widehat{z} \otimes \widehat{z}\right]^{\mathcal{S}}. \tag{37}$$

We here denote by a double overline the projection on the subspace orthogonal to $(\mathbf{I} - \widehat{z} \otimes \widehat{z})$ and $\widehat{z} \otimes \widehat{z}$. The first two terms in the r.h.s. of Eq. (37) do not contribute to $\boldsymbol{\pi}_r^{[1]}$ but has to be retained for the next corrective terms. The heat flux term $\nabla \cdot \mathbf{q}_r$ is to be kept at this order when dealing with weakly nonlinear magnetosonic waves but arises at the next order when dealing with Alfvén waves. It is estimated in Section IV. When $\nabla \cdot \mathbf{q}_r$ is neglected, one recovers the classical gyro-viscous tensor,[23–25]

$$\pi_{pxx}^{[1]} = -\pi_{pyy}^{[1]} = -\frac{p_{\perp p}}{2\Omega_p}(\partial_y u_x + \partial_x u_y) \tag{38}$$

$$\pi_{pzz}^{[1]} = 0 \tag{39}$$

$$\pi_{pxy}^{[1]} = \pi_{pyx}^{[1]} = -\frac{p_{\perp p}}{2\Omega_p}(\partial_y u_y - \partial_x u_x) \tag{40}$$

$$\pi_{pyz}^{[1]} = \pi_{pzy}^{[1]} = \frac{1}{\Omega_p}[2p_{\|p}\partial_z u_x + p_{\perp p}(\partial_x u_z - \partial_z u_x)] \tag{41}$$

$$\pi_{pxz}^{[1]} = \pi_{pzx}^{[1]} = -\frac{1}{\Omega_p}[2p_{\|p}\partial_z u_y + p_{\perp p}(\partial_y u_z - \partial_z u_y)], \tag{42}$$

here given for the protons (the electron contribution being negligible due to the large mass ratio) and usually obtained in an $1/\Omega_p$-expansion.

The next correction $\boldsymbol{\pi}_p^{[2]}$ originates from terms neglected in Eq.(31), together with the dominant contributions in Eq. (32). We can consistently write (replacing single overlines by double ones)

$$\boldsymbol{\pi}_p^{[2]} \times \widehat{z} - \widehat{z} \times \boldsymbol{\pi}_p^{[2]} = \overline{\overline{L(\boldsymbol{\pi}_p^{[1]})}} + \overline{\overline{\boldsymbol{\chi}_p^{[2]}}} + D[\boldsymbol{\chi}_p^{[1]}] + \left((\widehat{b} - \widehat{z}) \times \boldsymbol{\pi}_p^{[1]} \right)^{\mathcal{S}}, \tag{43}$$

together with the conditions

$$\widehat{z} \cdot \boldsymbol{\pi}_p^{[2]} \cdot \widehat{z} + \left((\widehat{b} - \widehat{z}) \cdot \boldsymbol{\pi}_p^{[1]} \cdot \widehat{z} \right)^{\mathcal{S}} = 0, \qquad \boldsymbol{\pi}_p^{[2]} : \mathbf{I} = 0. \tag{44}$$

At the order of the present approximation,

$$\overline{\overline{L(\boldsymbol{\pi}_p^{[1]})}} = \frac{1}{\Omega_p} \partial_t \boldsymbol{\pi}_p^{[1]}. \tag{45}$$

Furthermore,

$$D[\boldsymbol{\chi}_p^{[1]}] \equiv \overline{\boldsymbol{\chi}_p^{[1]}} - \overline{\overline{\boldsymbol{\chi}_p^{[1]}}} = \frac{1}{2}[(\widehat{b} - \widehat{z}) \otimes \widehat{z} + \widehat{z} \otimes (\widehat{b} - \widehat{z})] : \boldsymbol{\chi}_p^{[1]} \ (\mathbf{I} - 3\widehat{z} \otimes \widehat{z})$$
$$+ \frac{1}{2}(\mathbf{I} - 3\widehat{z} \otimes \widehat{z}) : \boldsymbol{\chi}_p^{[1]}[(\widehat{b} - \widehat{z}) \otimes \widehat{z} + \widehat{z} \otimes (\widehat{b} - \widehat{z})] \tag{46}$$

and

$$\boldsymbol{\chi}_p^{[2]} = \frac{1}{\Omega_p} \frac{d}{dt}(p_{\|p} - p_{\perp p}) \left((\widehat{b} - \widehat{z}) \otimes \widehat{z} \right)^{\mathcal{S}}$$
$$+ \frac{1}{\Omega_p}(p_{\|p} - p_{\perp p}) \Big[2(\widehat{b} - \widehat{z}) \cdot \nabla u \otimes \widehat{z} + 2\widehat{z} \cdot \nabla u \otimes (\widehat{b} - \widehat{z}) - 2(\widehat{z} \cdot \nabla u \cdot \widehat{z})(\widehat{b} - \widehat{z}) \otimes \widehat{z}$$
$$- \left((\widehat{b} - \widehat{z}) \cdot \nabla u \cdot \widehat{z} + \widehat{z} \cdot \nabla u \cdot (\widehat{b} - \widehat{z}) \right) \widehat{z} \otimes \widehat{z} + h \otimes \widehat{z} - (h \cdot \widehat{z})\widehat{z} \otimes \widehat{z} \Big]^{\mathcal{S}} \tag{47}$$

with

$$h = \frac{1}{\Omega_p} \nabla \times \left(\frac{1}{4\pi\rho}(b \times (\nabla \times b) - \frac{1}{\rho}\nabla \cdot \mathbf{P}_e^G \right). \tag{48}$$

All the terms in $D[\boldsymbol{\chi}_p^{[1]}]$ and in $\boldsymbol{\chi}_p^{[2]}$ with the exception of those involving h (that originates from the generalized Ohm's law) result from field line distortion and are only relevant for the scaling of oblique Alfvén waves. For such waves, $\widehat{z} \cdot (\widehat{b} - \widehat{z})$ is negligible, which enables

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one to write

$$\overline{\overline{\boldsymbol{\chi}_p^{[2]}}} + D[\boldsymbol{\chi}_p^{[1]}] = \frac{1}{\Omega_p}\left((\widehat{b} - \widehat{z}) \cdot \nabla u \cdot \widehat{z} + \widehat{z} \cdot \nabla u \cdot (\widehat{b} - \widehat{z})\right)\left[p_{\perp p}\mathbf{I} + (p_{\perp p} - 4p_{\|p})\widehat{z} \otimes \widehat{z}\right]$$
$$+ \frac{1}{\Omega_p}\left(p_{\perp p}\nabla \cdot u + (p_{\perp p} - 4p_{\|p})\widehat{z} \cdot \nabla u \cdot \widehat{z}\right)\left[(\widehat{b} - \widehat{z}) \otimes \widehat{z}\right]^{\mathcal{S}}$$
$$+ \frac{2}{\Omega_p}(p_{\perp p} - p_{\|p})\left[(\widehat{b} - \widehat{z}) \cdot \nabla u \otimes \widehat{z} + \widehat{z} \cdot \nabla u \otimes (\widehat{b} - \widehat{z})\right]^{\mathcal{S}}$$
$$+ \frac{1}{\Omega_p}(p_{\perp p} - p_{\|p})(h \otimes \widehat{z}). \tag{49}$$

This contribution is usually neglected,[26] and so are all the other terms in Eq. (43), except $\frac{1}{\Omega_p}\frac{\partial}{\partial t}\boldsymbol{\pi}_p^{[1]}$. Retaining the nonlinear terms originating from the field line distortion is nevertheless important to prevent the onset of spurious nonlinearities (making the problem ill-posed) in the equation governing the dynamics of weakly nonlinear oblique Alfvén waves.[17] These waves appear to be governed by a *linear* Korteweg de Vries equation with nonlocal damping. As in the Hall-MHD description, nonlinear couplings turn out to vanish.[27,28]

IV. MODELING OF THE HEAT FLUXES

It is again convenient in Eqs. (25) and (26), to separate the contributions to the gyrotropic part of $\nabla \cdot \mathbf{q}_r$, originating from the gyrotropic and non-gyrotropic heat fluxes, by writing $\mathbf{q}_r = \mathbf{q}_r^G + \mathbf{q}_r^{NG}$ with

$$q_{r\,ijk}^G = q_{\|r}\widehat{b}_i\widehat{b}_j\widehat{b}_k + q_{\perp r}(\delta_{ij}\widehat{b}_k + \delta_{ik}\widehat{b}_j + \delta_{jk}\widehat{b}_i - 3\widehat{b}_i\widehat{b}_j\widehat{b}_k). \tag{50}$$

The equations for the gyrotropic pressure components involve

$$\widehat{b} \cdot (\nabla \cdot \mathbf{q}_r^G) \cdot \widehat{b} = \nabla \cdot (\widehat{b}\,q_{\|r}) - 2q_{\perp r}\nabla \cdot \widehat{b} \tag{51}$$
$$\frac{1}{2}\left(\mathrm{tr}(\nabla \cdot \mathbf{q}_r^G) - \widehat{b} \cdot (\nabla \cdot \mathbf{q}_r^G) \cdot \widehat{b}\right) = \nabla \cdot (\widehat{b}\,q_{\perp r}) + q_{\perp r}\nabla \cdot \widehat{b}, \tag{52}$$

together with the contribution of the non-gyrotropic heat fluxes to the gyrotropic part of $\nabla \cdot \mathbf{q}_r$ that we denote $(\nabla \cdot \mathbf{q}_r^{NG})^G$. The non-gyrotropic part of $\nabla \cdot \mathbf{q}_r$ contributes to the non-gyrotropic pressure corrections.

A. Non-gyrotropic heat flux contributions

The gyrotropic heat flux contributions are comparable to the pressure perturbations (as seen from the gyrotropic pressure equations), i.e. of order $\epsilon^{1/2}$ for parallel Alfvén waves

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and of order ϵ for oblique Alfvén and magnetosonic waves. On the other hand, the non-gyrotropic heat flux components do not only behave like the product of a pressure and a velocity but also involve an additional space derivative arising together with the $1/\Omega_r$ factor. From the scaling assumptions, one can conclude that these contributions are subdominant for both parallel Alfvén and oblique magnetosonic waves, while they are of the same order as the gyrotropic heat flux components in the case of oblique Alfvén waves. This observation is confirmed by the kinetic theory based on Vlasov-Maxwell equations (see refs.[16,17] and Appendix B).

The non-gyrotropic heat fluxes obtained for oblique Alfvén waves (Appendices B and C) can be expressed in terms of current $j = \dfrac{c}{4\pi}\nabla \times b$ and diamagnetic drifts of each particle species $u_{d,r} = \dfrac{c}{nq|b|^2}b \times \nabla \cdot \mathbf{p}_r$ that in the considered limit are given by $\dfrac{j}{qn} \approx \dfrac{v_A^2}{\Omega_p}\left(-\partial_z\dfrac{b_y}{B_0}, 0, \partial_x\dfrac{b_y}{B_0}\right)$ and $u_{d,r} \approx \dfrac{v_{\Delta r}^2}{\Omega_p}\left(\partial_z\dfrac{b_y}{B_0}, 0, 0\right)$, where we define the squared Alfvén velocity $v_A^2 = \dfrac{B_0^2}{4\pi\rho^{(0)}}$ and also $v_{\Delta r}^2 = \dfrac{p_{\perp r}^{(0)} - p_{\parallel r}^{(0)}}{\rho^{(0)}}$. We thus infer the closure approximation

$$(\nabla \cdot \mathbf{q}_e^{NG})^G = 2\nabla_\perp \cdot [p_{\perp e}(u_{d,e} - \dfrac{j}{qn})](I - \widehat{b} \otimes \widehat{b}) + \nabla_\perp \cdot [p_{\parallel e}(u_{d,e} - \dfrac{j}{qn})]\widehat{b} \otimes \widehat{b} \quad (53)$$

$$(\nabla \cdot \mathbf{q}_p^{NG})^G = 2\nabla_\perp \cdot [p_{\parallel p}u_{d,p}]\widehat{b} \otimes \widehat{b}, \quad (54)$$

and also approximate the non-gyrotropic part of $\nabla \cdot \mathbf{q}_p$ by

$$\overline{\overline{\nabla \cdot \mathbf{q}_p}} = \dfrac{p_{\perp p}}{2}(\nabla_\perp \otimes u_{dp} - (\widehat{z} \times \nabla_\perp) \otimes (\widehat{z} \times u_{dp}))$$
$$+ \left[\widehat{z} \otimes \left(\nabla_\perp q_{\perp p} - p_{\perp p}\dfrac{v_{\Delta p}^2}{2\Omega_p}\widehat{z} \times \Delta_\perp\widehat{b} - 2p_{\parallel p}\widehat{z} \times (\nabla \times u_{dp})\right)\right]^{\mathcal{S}}. \quad (55)$$

B. Gyrotropic heat fluxes

In order to infer closed expressions for the gyrotropic heat fluxes, based on the predictions of the kinetic theory for oblique Alfvén waves (Appendix C), we adapt the approach developed in the context of parallel propagation[12] where the closure approximations eventually reduce to the replacement of the plasma response function W_r by its two or four-pole Padé approximants.

1. Parallel heat flux

In order to reduce the problem to a form close to that of parallel propagation, starting from Eq. (C.11) where quantities proportional to m_e/m_p have been neglected, we first define

$$\frac{q'_{\|r}}{v_{th,r}p_{\|r}^{(0)}} = \frac{q_{\|r}}{v_{th,r}p_{\|r}^{(0)}} - 3\left(\frac{v_{\Delta e}^2 + v_A^2}{v_A^2}\right)\left(\frac{\Omega_p}{\Omega_r} - 1\right)\frac{j_\|}{nqv_{th,r}}, \tag{56}$$

where we used the relation $c_r^2 = (v_A^2 + v_{\Delta e}^2 + v_{\Delta p}^2)/v_{th,r}^2$ and the expression of the parallel current in the asymptotics of long oblique Alfvén waves given in the previous sub-section. It follows that

$$\frac{q'_{\|r}}{v_{th,r}p_{\|r}^{(0)}} = c_r\frac{c_r^2 - 3 + W_r^{-1}}{c_r^2 - 1 + W_r^{-1}}\frac{T_{\|r}^{(1)}}{T_{\|r}^{(0)}}, \tag{57}$$

where $T_{\|r}^{(1)}$ denotes the parallel temperature perturbations for the particles of species r and $T_{\|r}^{(0)}$ the corresponding equilibrium value. Similar notations are used for the transverse temperatures. We then proceed as in Ref.[16]. The parameter c_r defined as the ratio of the phase velocity projected on the direction of the ambient field to the thermal velocity of species r, is now more generally viewed as the ratio $X = -\frac{1}{v_{th,r}}\partial_t\partial_z^{-1}$. The operator $\mathcal{F}_\|(X)$ defined by

$$\frac{q'_{\|r}}{v_{th,r}p_{\|r}^{(0)}} = \mathcal{F}_\|(X)\frac{T_{\|r}^{(1)}}{T_{\|r}^{(0)}} \tag{58}$$

is approximated by a homographic function

$$\mathcal{F}_\|(X) = (Q_\|^3 + Q_\|^4 X\mathcal{H})^{-1}(Q_\|^1 X + Q_\|^2\mathcal{H}). \tag{59}$$

where $\mathcal{H}$ is the Hilbert transform with respect to the parallel coordinate z, which allows one to eventually get a first order initial value problem. The constant coefficients $Q_\|^i$ are chosen in a way that ensures the correct asymptotic behavior of the parallel heat fluxes in both the isothermal and adiabatic limits. As shown by SHD, this prescription results in a satisfactory modeling in the intermediate regimes. In the isothermal limit ($c_r \ll 1$), $W_r \approx 1 - c_r^2 + \sqrt{\frac{\pi}{2}}c_r\mathcal{H}_\xi$ and $q'^{(1)}_{\|r} = -\sqrt{\frac{8}{\pi}}v_{th,r}n^{(0)}\mathcal{H}_\xi T_{\|r}^{(1)}$ independent of c_r. Differently, in the adiabatic limit ($c_r \gg 1$), $W_r \approx -1/c_r^2 - 3/c_r^4 - 15/c_r^6$ and the heat fluxes are negligible. One thus gets $Q_\|^1 = 0$, $Q_\|^2 = -\sqrt{\frac{8}{\pi}}$, $Q_\|^3 = 1$, $Q_\|^4 = -\sqrt{\frac{8}{\pi}}(\frac{3\pi}{8} - 1)$. In this approximation, the corrected parallel heat flux $q'_{\|r}$ is thus determined in terms of the parallel temperature

$T_{\|r}$ by the partial differential equation

$$\left(\frac{d}{dt} + \frac{v_{th,r}}{\sqrt{\frac{8}{\pi}(1 - \frac{3\pi}{8})}}\mathcal{H}\partial_z\right)\frac{q'_{\|r}}{v_{th,r}p^{(0)}_{\|r}} = \frac{1}{1 - \frac{3\pi}{8}}v_{th,r}\partial_z\frac{T_{\|r}}{T^{(0)}_{\|r}} \tag{60}$$

where, to restore Galilean invariance, the convective derivative $\partial_t + u \cdot \nabla$ has been substituted to the partial time derivative.

2. *Perpendicular heat flux*

Proceeding in a similar way, starting from Eq. (C.12) or (C.13), we first define

$$\frac{q^\star_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} = \frac{q_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} + \left[\left(1 + \frac{v^2_{\Delta e} + v^2_{\Delta p}}{v^2_A}\right)\left(\frac{\Omega_p}{\Omega_r} + 1\right) - \frac{v^2_{\Delta p}}{v^2_A} + 2\frac{v^2_{\Delta r}}{v^2_A}\frac{\Omega_p}{\Omega_r}\right]\frac{j_\|}{nqv_{th,r}} \tag{61}$$

that, for long oblique Alfvén waves, can be expressed either in terms of $A = \dfrac{b_z}{B_0} + \dfrac{b^2_y}{2B^2_0}$ or in terms of $\left[\dfrac{T^{(1)}_{\perp r}}{T^{(0)}_{\perp r}} - \dfrac{3}{v_{th,r}}\partial_t\partial_z^{-1}\dfrac{\Omega_p}{\Omega_r}\dfrac{v^2_{th,r}}{v^2_A}\dfrac{j_\|}{nqv_{th,r}}\right]$, by means of operators that as previously are to be approximated. In order to accurately fit the adiabatic and isothermal limits, it is convenient to use a mixed expression involving both dependencies, in the form

$$\frac{q^\star_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} = \mathcal{F}^1_\perp\left(-\frac{1}{v_{th,r}}\partial_t\partial_z^{-1}\right)\left[\frac{T^{(1)}_{\perp r}}{T^{(0)}_{\perp r}} - \frac{3}{v_{th,r}}\partial_t\partial_z^{-1}\frac{\Omega_p}{\Omega_r}\frac{v^2_{th,r}}{v^2_A}\frac{j_\|}{nqv_{th,r}}\right] + \mathcal{F}^2_\perp\left(-\frac{1}{v_{th,r}}\partial_t\partial_z^{-1}\right)A. \tag{62}$$

Prescribing again a homographic form for the operators

$$\mathcal{F}^1_\perp(X) = (Q^3_\perp + Q^4_\perp X\mathcal{H})^{-1}(Q^1_\perp X + Q^2_\perp\mathcal{H}) \tag{63}$$

$$\mathcal{F}^2_\perp(X) = (Q^3_\perp + Q^4_\perp X\mathcal{H})^{-1}(Q^5_\perp X + Q^6_\perp\mathcal{H}), \tag{64}$$

we are led to choose $Q^1_\perp = Q^5_\perp = 0$, $Q^2_\perp = Q^4_\perp = -\sqrt{\dfrac{2}{\pi}}$ and $Q^6_\perp = \sqrt{\dfrac{2}{\pi}}\left(1 - \dfrac{T^{(0)}_{\perp r}}{T^{(0)}_{\|r}}\right)$ and get

$$\left(\frac{\partial}{\partial t} - \sqrt{\frac{\pi}{2}}v_{th,r}\mathcal{H}\partial_z\right)\frac{q^\star_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} = v_{th,r}\partial_z\left(\left(1 - \frac{T^{(0)}_{\perp r}}{T^{(0)}_{\|r}}\right)\frac{|b|}{B_0} - \left(\frac{T_{\perp r}}{T^{(0)}_{\perp r}} - \frac{3v_{th,r}}{v^2_A}\frac{\Omega_p}{\Omega_r}\partial_t\partial_z^{-1}\frac{j_\|}{nqv_{th,r}}\right)\right). \tag{65}$$

Introducing

$$\frac{q'_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} = \frac{q^\star_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} - 3\frac{v^2_{th,r}}{v^2_A}\frac{\Omega_p}{\Omega_r}\frac{j_\|}{nqv_{th,r}},$$

$$= \frac{q_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} + \left[\left(1 + \frac{v^2_{\Delta e} + v^2_{\Delta p}}{v^2_A}\right)\left(\frac{\Omega_p}{\Omega_r} + 1\right) - \frac{v^2_{\Delta p}}{v^2_A} + \frac{2v^2_{\Delta r} - 3v^2_{th,r}}{v^2_A}\frac{\Omega_p}{\Omega_r}\right]\frac{j_\|}{nqv_{th,r}}, \tag{66}$$

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we finally obtain

$$
\left(\frac{d}{dt} - \sqrt{\frac{\pi}{2}} v_{th,r} \mathcal{H} \partial_z\right) \frac{q'_{\perp r}}{v_{th,r} p^{(0)}_{\perp r}} = v_{th,r} \partial_z \left(\left(1 - \frac{T^{(0)}_{\perp r}}{T^{(0)}_{\parallel r}}\right) \frac{|b|}{B_0} - \frac{T_{\perp r}}{T^{(0)}_{\perp r}} + 3\sqrt{\frac{\pi}{2}} \frac{v^2_{th,r}}{v^2_A} \frac{\Omega_p}{\Omega_r} \mathcal{H} \frac{j_\parallel}{nqv_{th,r}}\right).
\tag{67}
$$

As previously, a convective derivative has been substituted to the partial time derivatives in order to restore Galilean invariance.

V.   THE MODEL AND ITS VALIDATION

A mono-fluid model has thus been constructed. It is defined by the closed system formed by Eqs. (9), (10), (16), (25), (26) where the s_{ir} terms are neglected and where Eqs. (51), (52), (53) and (54) have been used, supplemented by Eqs. (56), (60), (66), (67), together with the non-gyrotropic pressure corrections $\boldsymbol{\pi}_r = \boldsymbol{\pi}^{[1]}_r + \boldsymbol{\pi}^{[2]}_r$ that are computed in Section III.B and involve the non-gyrotropic heat flux given by Eq. (55).

To validate this model, we consider its predictions for the various MHD waves in the long-wavelength limit. Our previous model[12] specifically designed to describe parallel Alfvén waves is easily recovered by prescribing the ordering associated to these waves. The non-gyrotropic heat flux contributions then disappear and only the leading order gyro-viscous tensor without the heat flux term is to be retained. In this regime, a reductive perturbative expansion leads to a generalized kinetic derivative nonlinear Schrödinger equation that identifies with that derived from the Vlasov-Maxwell system,[16] up to the replacement of the plasma response function by its two or four-pole Padé approximants.[12]

The model derived in the present paper is in contrast needed to describe oblique Alfvén waves. We demonstrate in this section that the kinetic theory presented in Appendix C is accurately reproduced, and so are the classical dispersion relations and Landau damping rates of oblique and kinetic Alfvén waves.

Denoting by ξ the coordinate along the direction of propagation, one has $\nabla = (\sin\alpha\, \partial_\xi, 0, \cos\alpha\, \partial_\xi)$ and $\partial_t = -V_0 \partial_\xi$ with $V_0 = \Lambda_0 \cos\alpha$. From Eq. (60), one immediately gets to leading order

$$
\frac{q'_{\parallel r}}{v_{th,r} p^{(0)}_{\parallel r}} = \frac{-\sqrt{\frac{8}{\pi}}\mathcal{H}}{1 - \sqrt{\frac{8}{\pi}}\left(\frac{3\pi}{8} - 1\right)c_r \mathcal{H}} \frac{T^{(1)}_{\parallel r}}{T^{(0)}_{\parallel r}} = c_r \frac{c^2_r - 3 + W^{-1}_{4r}}{c^2_r - 1 + W^{-1}_{4r}} \frac{T^{(1)}_{\parallel r}}{T^{(0)}_{\parallel r}},
\tag{68}
$$

that for the protons and the electrons respectively give

$$\frac{q_{\|p}^{(1)}}{v_{th,p}p_{\|p}^{(0)}} = c_p \frac{c_p^2 - 3 + W_{4p}^{-1}}{c_p^2 - 1 + W_{4p}^{-1}} \frac{T_{\|p}^{(1)}}{T_{\|p}^{(0)}} \tag{69}$$

$$\frac{q_{\|e}^{(1)}}{v_{th,p}p_{\|p}^{(0)}} = c_e \frac{c_e^2 - 3 + W_{4e}^{-1}}{c_e^2 - 1 + W_{4e}^{-1}} \frac{T_{\|e}^{(1)}}{T_{\|e}^{(0)}} - 3\frac{v_A^2 + v_{\Delta e}^2}{v_{th,e}} \frac{\sin\alpha}{\Omega_p} \partial_\xi \frac{b_y^{(1)}}{B_0}. \tag{70}$$

This reproduces Eq. (C.11), up to the replacement of the plasma response function W_r by its four-pole approximant defined as

$$W_{4r} = \frac{\frac{1}{2}(8 - 3\pi)c_r^2 - \sqrt{2\pi}c_r\mathcal{H} + 4}{\frac{1}{2}c_r^4(3\pi - 8) + \sqrt{2\pi}c_r^3\mathcal{H} + \frac{1}{2}(16 - 9\pi)c_r^2 - 3\sqrt{2\pi}c_r\mathcal{H} + 4}. \tag{71}$$

Substituting in the equation for the parallel proton pressure that rewrites

$$\frac{p_{\|p}^{(1)}}{p_{\|p}^{(0)}} - 3\frac{n^{(1)}}{n^{(0)}} + 2A = \frac{2}{\Lambda_0} \frac{v_A^2 + v_{\Delta e}^2 + v_{\Delta p}^2}{\Omega_p} \sin\alpha\partial_\xi \frac{b_y^{(1)}}{B_0} + \frac{1}{c_r} \frac{q_{\|p}^{(1)}}{v_{th,p}p_{\|p}^{(0)}}, \tag{72}$$

one gets

$$\frac{p_{\|p}^{(1)}}{p_{\|p}^{(0)}} = (c_p^2 + W_{4p}^{-1})\frac{n^{(1)}}{n^{(0)}} - (c_p^2 - 1 + W_{4p}^{-1})A + (c_p^2 - 1 + W_{4p}^{-1})\frac{\Lambda_0}{\Omega_p} \sin\alpha\partial_\xi \frac{b_y^{(1)}}{B_0} \tag{73}$$

$$\frac{p_{\|e}^{(1)}}{p_{\|e}^{(0)}} - 3\frac{n^{(1)}}{n^{(0)}} + 2A = \frac{c_e^2 - 3 + W_{4e}^{-1}}{c_e^2 - 1 + W_{4e}^{-1}} \left(\frac{p_{\|e}^{(1)}}{p_{\|e}^{(0)}} - \frac{n^{(1)}}{n^{(0)}}\right) \tag{74}$$

that correspond to Eq. (C.6).

From the transverse heat flux equations, we get

$$\frac{q_{\perp e}^{(1)}}{v_{th,e}p_{\perp e}^{(0)}} = \frac{-1}{c_e + \sqrt{\frac{\pi}{2}}\mathcal{H}} \left[\left(1 - \frac{T_{\perp e}^{(0)}}{T_{\|e}^{(0)}}\right)A - \frac{p_{\perp e}^{(1)}}{p_{\perp e}^{(0)}} + \frac{n^{(1)}}{n^{(0)}}\right] - \frac{v_A^2 + v_{\Delta e}^2}{v_{th,e}} \frac{\sin\alpha}{\Omega_p} \partial_\xi \frac{b_y^{(1)}}{B_0}. \tag{75}$$

Substituting in the equation for the electron parallel pressure that rewrites

$$\frac{p_{\perp e}^{(1)}}{p_{\perp e}^{(0)}} = \frac{n^{(1)}}{n^{(0)}} + A + \frac{v_A^2 + v_{\Delta e}^2}{\Lambda_0} \frac{\sin\alpha}{\Omega_p} \partial_\xi \frac{b_y^{(1)}}{B_0} + \frac{q_{\perp e}^{(1)}}{\Lambda_0 p_{\perp e}}, \tag{76}$$

one obtains

$$\frac{p_{\perp e}^{(1)}}{p_{\perp e}^{(0)}} = \frac{n^{(1)}}{n^{(0)}} + \left(1 - \frac{T_{\|e}^{(0)}}{T_{\|e}^{(0)}}W_{2e}\right)A. \tag{77}$$

Here,

$$W_{2r} = \frac{1}{1 - \sqrt{\frac{\pi}{2}}c_r\mathcal{H} - c_r^2} \tag{78}$$

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is the two-pole approximant of the plasma response function $\mathcal{W}_r$.

Similarly, for the protons

$$\frac{p_{\perp p}^{(1)}}{p_{\perp p}^{(0)}} = \frac{n^{(1)}}{n^{(0)}} + A - \frac{v_A^2 + v_{\Delta e}^2}{\Lambda_0}\frac{\sin\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0} + \frac{q_{\perp p}^{(1)}}{\Lambda_0 p_{\perp p}} \tag{79}$$

with

$$\frac{q_{\perp p}^{(1)}}{v_{th,p}p_{\perp p}^{(0)}} = \frac{-1}{c_p + \sqrt{\frac{\pi}{2}}\mathcal{H}}\left[\left(1 - \frac{T_{\perp p}^{(0)}}{T_{\parallel p}^{(0)}}\right)A - \frac{p_{\perp p}^{(1)}}{p_{\perp p}^{(0)}} + \frac{n^{(1)}}{n^{(0)}} + 3\sqrt{\frac{\pi}{2}}v_{th,p}\mathcal{H}\frac{\sin\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0}\right]$$
$$+\left(3(v_{th,p}^2 - v_{\Delta p}^2) - 2(v_A^2 + v_{\Delta e}^2)\right)\frac{\sin\alpha}{v_{th,p}\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0}, \tag{80}$$

which implies

$$\frac{p_{\perp p}^{(1)}}{p_{\perp p}^{(0)}} = \frac{n^{(1)}}{n^{(0)}} + \left(1 - \frac{T_{\perp p}^{(0)}}{T_{\parallel p}^{(0)}}W_{2p}\right)A - 3\Lambda_0\frac{\sin\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0}. \tag{81}$$

Again the result of the kinetic theory, as given by Eq. (C.8), is recovered up to the replacement of the plasma response function by a Padé approximant.

To push further the validation of the present model, it is of interest to concentrate on the regime $\frac{m_e}{m_i} \ll \beta \ll \frac{T_e}{T_p}$, with $\beta = \frac{1}{v_A^2}\frac{T_e}{m_p}$, assuming no temperature anisotropy for easier comparison with classical results. This ordering corresponds to the limit $c_e \to 0$ of isothermal electrons and $c_p \to \infty$ of adiabatic protons.

In the limit $c_e \to 0$, $W_{4e} \approx W_{2e} \approx W_e \approx 1 - c_e^2 + \sqrt{\frac{\pi}{2}}c_e\mathcal{H}$ and we get

$$\frac{p_{\parallel e}^{(1)}}{\rho^{(0)}} = \beta v_A^2\frac{p_{\parallel e}^{(1)}}{p_{\parallel e}^{(0)}} = v_A^2\left\{\left[\beta - \sqrt{\frac{\pi}{2}}\sqrt{\beta}\sqrt{\frac{m_e}{m_p}}\mathcal{H}\right]\frac{\rho^{(1)}}{\rho^{(0)}} + \sqrt{\beta}\sqrt{\frac{\pi}{2}}\sqrt{\frac{m_e}{m_p}}\mathcal{H}A\right\} \tag{82}$$

$$\frac{p_{\perp e}^{(1)}}{\rho^{(0)}} = \beta v_A^2\frac{p_{\perp e}^{(1)}}{p_{\perp e}^{(0)}} = v_A^2\left[\beta\frac{\rho^{(1)}}{\rho^{(0)}} - \sqrt{\beta}\sqrt{\frac{\pi}{2}}\sqrt{\frac{m_e}{m_p}}\mathcal{H}A\right], \tag{83}$$

that provides a systematic derivation of relations previously based on a phenomenological argument.[17]

The adiabatic limit $c_p \to \infty$ assumes that the phase velocity of the wave is much larger than the thermal velocity, which is not consistent with the long-wave asymptotics. The adiabatic limit is thus conveniently taken by prescribing zero heat fluxes, and the relations (60)-(61) of Ref.[17] are then immediately recovered. By inspection, it is also easily verified that the gyro-viscous tensor defined by Eq. (43) identifies within the reductive perturbative scaling with Eqs. (B2)-(B7) of Ref.[17]. In particular Eq. (49) reproduces Eqs. (B15)-(B18)

of the same reference. The remaining of the asymptotic analysis is straightforward and is performed in Ref.[17]. Direct comparisons are successfully made with kinetic results[18,19]. This demonstrates that the present model correctly reproduces the dynamics of small amplitude oblique and kinetic Alfvén waves.

The case of oblique magneto-sonic waves requires a more detailed discussion that we explicit in the regime of adiabatic ions and isothermal electrons with isotropic temperatures. The leading order linear dispersion relation correctly reproduces that provided by the kinetic theory (see Eqs. (29) and (A.21) of Ref.[17]). It includes a Landau damping rate that, up to an angular factor, scales like $k v_A \sqrt{\beta} \sqrt{\frac{m_e}{m_p}}$ where, as above, β is defined as the ratio of the electron to magnetic pressures and k the wavenumber of the perturbation. This level of description is sufficient when this rate of damping is larger than the inverse nonlinear time ku (where u is a typical velocity perturbation), that is to say when $\sqrt{\beta} \sqrt{\frac{m_e}{m_p}} \gg \epsilon \frac{V_0}{v_A}$. The parameter ϵ can be estimated as $k l_p$, where $l_p = \frac{v_A}{\Omega_p}$ is the proton inertial length. Let us first consider the distinguished limit where the wave amplitude scales like ϵ. For slow waves for which $V_0 \sim \sqrt{\beta} v_A$, the condition reduces to $k l_p \ll \left(\frac{m_e}{m_p}\right)^{1/4} \approx 0.15$. These waves are thus strongly damped in the long-wave limit. For fast waves $V_0 \sim v_A$, and the condition for rapid damping reads $\beta \gg \frac{m_p}{m_e}(k l_p)^4$. When this condition is not satisfied, the Landau damping arises at the same order as the nonlinear and dispersive terms and a weakly nonlinear analysis on the time scale $\tau = \epsilon^{3/2}$ is required. In this regime, the equations for $\partial_\tau u_x^{(1)}$ and $\partial_\tau u_z^{(1)}$ involve the quantities $\partial_x p_\perp^{(2)}$ and $\partial_z p_\parallel^{(2)}$, and thus the gyrotropic heat fluxes $q_\perp^{(2)}$ and $q_\parallel^{(2)}$, together with the FLR term $\boldsymbol{\pi}_r^{[2]}$ that through Eq. (43) is prescribed by $\frac{1}{\Omega_p}(\partial_t \boldsymbol{\pi}_r^{[1]} + \overline{\overline{\nabla \cdot \mathbf{q}_p^{(3/2)}}})$. These heat fluxes, when not negligible, are not properly modeled in the present formalism. They are absent in the case of purely transverse propagation, a situation addressed in Ref.[29]. The case of oblique propagation in the adiabatic limit was considered in[30] where the term $\frac{1}{\Omega_p} \partial_t \boldsymbol{\pi}_r^{[1]}$ was overlooked.

When the amplitude is larger, usual MHD supplemented by $1/\Omega_p$ corrections provides a sufficient description. As the amplitude of these waves is reduced by dissipation, the regime dominated by Landau damping is recovered. The only case where our model does not provide a complete description of magnetosonic waves thus concerns small amplitude waves with the distinguished scaling and very small β.

The question arises whether the usual energy $E = \int \left(\rho \frac{u^2}{2} + \frac{b^2}{8\pi} + p_\perp + \frac{1}{2} p_\parallel \right) d^3 x$ is conserved by the above mono-fluid model. The delicate contributions originate from the electron pressure gradient in the induction equation and from the second order non-gyrotropic pressure corrections. The first term that affects the magnetic field evolution only in the case of pressure anisotropy, contributes in a long wave theory at the level of the linear dispersion relation. In this limit, it can thus be replaced by $\frac{1}{\rho_0} \nabla p_{\perp e} - \frac{v_{\Delta e}^2}{B_0^2} \nabla \cdot (b \otimes b)$, a term that does not contribute to the energy budget. Concerning the non-gyrotropic pressure contributions, while the leading order $\boldsymbol{\pi}^{[1]}$ preserves energy (at least in the absence of the heat flux term), the effect of $\boldsymbol{\pi}^{[2]}$ is still unsettled. This question requires further investigations. In fact, in a way similar to the diamagnetic term in the generalized Ohm's law, this contribution is only relevant at the level of the linear dispersion relation of oblique and kinetic Alfvén waves. As a consequence, even in the case where the energy is only conserved at the order of validity of the performed approximations, the effect on the large-scale dynamics will be negligible.

VI. CONCLUSION

A mono-fluid model has been derived with the constraint to reproduce the weakly nonlinear dynamics and the Landau damping of long MHD waves in a collisionless plasma, for any β larger than the electron to proton mass ratio and any angle of propagation. It reproduces the dynamics of small-amplitude oblique Alfvén waves, including the exact cancellation of the nonlinearity. For parallel Alfvén waves, it leads to the KDNLS equation and describes the transverse instability of a circularly polarized wave,[31] resulting in the formation of intense magnetic filaments. This Alfvén wave "collapse" was considered as a possible mechanism at the origin of the cylindrical field aligned current tubes observed by the CLUSTER mission in the terrestrial magnetosheath.[32]

Comparison of the model with gyrokinetic simulations and possibly with Vlasov-Maxwell or particles in cells simulations, in particular in the nonlinear stage of parametric instabilities, are in project. It is also necessary to evaluate the importance of particle trapping that requires a nonlinear fluid closure, presently very difficult to design.[33]

This model can be used to perform three-dimensional numerical simulations of dispersive MHD turbulence, taking into account realistic dissipation and heating mechanisms. The retained second-order FLR corrections should in particular provide an accurate description

of the kinetic Alfvén waves generated at small scales. Such simulations, that involve a self-consistent treatment of the turbulent dynamics in the presence of Landau damping, could significantly contribute to the understanding of cosmic ray scattering (by fast and Alfvén modes) in the interstellar medium.[34]

This model will also be useful to study the formation of coherent structures such as magnetic holes[35,36], shocklets and also the structures resulting from the nonlinear evolution of the mirror instability, observed in the solar wind[37] and the magnetosheath.[38] A correct description of this instability that extends up to scales comparable to the ion Larmor radius requires higher order FLR corrections that, as mentioned in Section III.B, cannot be directly obtained within a monofluid description. Their evaluation is possible through a $1/\Omega_p$ expansion of all the fields in a multi-fluid description.[43]

Another development concerns hybrid simulations that could possibly be improved by replacing the usual MHD description of the electron dynamics by a more refined one including physical processes retained by the present model.

APPENDIX A: LONG-WAVE EXPANSION OF VLASOV-MAXWELL EQUATIONS FOR OBLIQUE ALFVÉN WAVES

We write the Vlasov-Maxwell equations in the form

$$\partial_t f_r + v \cdot \nabla f_r + \frac{q_r}{m_r}(e + \frac{1}{c}v \times b) \cdot \nabla_v f_r = 0 \tag{A.1}$$

$$\frac{1}{c}\partial_t b = -\nabla \times e \tag{A.2}$$

$$\nabla \times b = \frac{4\pi}{c}\sum_r q_r n_r \int v f_r d^3v + \frac{1}{c}\partial_t e \tag{A.3}$$

$$\nabla \cdot e = 4\pi \sum_r q_r n_r \int f_r d^3v, \tag{A.4}$$

where f_r and n_r are the distribution function and the average number density of the particles of species r with charge q_r and mass m_r. The displacement current $\frac{1}{c}\partial_t e$ turns out to be negligible in the present analysis where perturbations propagate at a velocity small compared with the speed of light. This contribution which might be important for auroral plasmas is retained by Verheest.[39]

Let α be the angle between the ambient magnetic field $B_0\hat{z}$ (where $\hat{z}$ is the unit vector pointing along the z-axis) and the direction of propagation of the wave. It is then convenient to perform the change of frame $x' = x\cos\alpha - z\sin\alpha$, $z' = x\sin\alpha + z\cos\alpha$, the dynamics being assumed independent of the y variable. We then introduce the stretched variable $\xi = \epsilon^{1/2}(z' - V_0 t)$ where $V_0 \ll c$ is the Alfvén-wave propagation velocity in the z direction, together with the slow time $\tau = \epsilon^{3/2}t$. It follows that the spatial gradient rewrites $\nabla = (\epsilon^{1/2}\sin\alpha\,\partial_\xi\,,\,0\,,\,\epsilon^{1/2}\cos\alpha\,\partial_\xi)$.

In order to select oblique Alfvén waves, we expand

$$b_x = \epsilon(b_x^{(1)} + \epsilon b_x^{(2)} + \cdots) \tag{A.5}$$

$$b_y = \epsilon^{1/2}(b_y^{(1)} + \epsilon b_y^{(2)} + \cdots) \tag{A.6}$$

$$b_z = B_0 + \epsilon(b_z^{(1)} + \epsilon b_z^{(2)} + \cdots). \tag{A.7}$$

and thus, from Eq. (A.2),

$$e_x = \epsilon^{1/2}(e_x^{(1)} + \epsilon e_x^{(2)} + \cdots) \tag{A.8}$$

$$e_y = \epsilon(e_y^{(1)} + \epsilon e_y^{(2)} + \cdots) \tag{A.9}$$

$$e_z = \epsilon^{1/2}(e_z^{(1)} + \epsilon e_z^{(2)} + \cdots), \tag{A.10}$$

with

$$\frac{V_0}{c}b_x^{(1)} = -\cos\alpha e_y^{(1)} \tag{A.11}$$

$$\frac{V_0}{c}b_y^{(1)} = \cos\alpha e_x^{(1)} - \sin\alpha e_z^{(1)} \tag{A.12}$$

$$\frac{V_0}{c}b_z^{(1)} = \sin\alpha e_y^{(1)}. \tag{A.13}$$

We also expand the distribution function in the form

$$f_r = F_r^{(0)} + \epsilon^{1/2}f_r^{(0)} + \epsilon f_r^{(1)} + \epsilon^{3/2}f_r^{(2)} + \cdots \tag{A.14}$$

where $F_r^{(0)}$ denotes the equilibrium velocity distribution function, assumed rotationally symmetric around the direction of the ambient field and symmetric relatively to forward

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and backward velocities along this direction, thus excluding the presence of equilibrium drifts.[39–41]

It is also convenient to express the velocity v in a cylindrical coordinate system by defining the azimuthal angle $\phi = \tan^{-1}(v_z/v_y)$ of the velocity component transverse to the ambient magnetic field. One writes

$$v = (v_x = v_\perp \cos \phi , v_y = v_\perp \sin \phi , v_z = v_\parallel) \tag{A.15}$$

and

$$\nabla_v = (\cos \phi\, \partial_{v_\perp} - \frac{\sin \phi}{v_\perp}\partial_\phi , \sin \phi\, \partial_{v_\perp} + \frac{\cos \phi}{v_\perp}\partial_\phi , \partial_{v_\parallel}) \tag{A.16}$$

Furthermore $\dfrac{q_r}{cm_r}(v \times B_0\widehat{z}) \cdot \nabla_v = -\Omega_r\partial_\phi$, where $\Omega_r = \dfrac{q_r B_0}{m_r c}$ is the gyro-frequency of the particles of species r.

Expanding to the successive orders, one gets from Eq. (A.1),

$$\Omega_r\partial_\phi F_r^{(0)} = 0 \tag{A.17}$$

$$\Omega_r\partial_\phi f_r^{(0)} = \frac{q_r}{m_r}\Sigma_1^{(1)} F_r^{(0)} \tag{A.18}$$

$$\Omega_r\partial_\phi f_r^{(1)} = \frac{q_r}{m_r}\left(\Sigma_2^{(1)} F_r^{(0)} + \Sigma_1^{(1)} f_r^{(0)}\right) + \Sigma_3 f_r^{(0)} \tag{A.19}$$

$$\Omega_r\partial_\phi f_r^{(2)} = \frac{q_r}{m_r}\left(\Sigma_1^{(2)} F_r^{(0)} + \Sigma_2^{(1)} f_r^{(0)} + \Sigma_1^{(1)} f_r^{(1)}\right) + \Sigma_3 f_r^{(1)} \tag{A.20}$$

where

$$\Sigma_1^{(s)} = (e_x^{(s)} - \frac{v_z}{c}b_y^{(s)})(\cos \phi\, \partial_{v_\perp} - \frac{\sin \phi}{v_\perp}\partial_\phi) + (e_z^{(s)} + \frac{v_x}{c}b_y^{(s)})\partial_{v_\parallel} \tag{A.21}$$

$$\Sigma_2^{(s)} = \frac{v_y}{c}b_z^{(s)}(\cos \phi\, \partial_{v_\perp} - \frac{\sin \phi}{v_\perp}\partial_\phi)$$

$$+ (e_y^{(s)} + \frac{v_z b_x^{(s)} - v_x b_z^{(s)}}{c})(\sin \phi\, \partial_{v_\perp} + \frac{\cos \phi}{v_\perp}\partial_\phi) - \frac{v_y b_x^{(s)}}{c}\partial_{v_\parallel} \tag{A.22}$$

$$\Sigma_3 = (v_x \sin \alpha + v_z \cos \alpha - V_0)\partial_\xi. \tag{A.23}$$

Equation (A.17) indicates that $F_r^{(0)}$ is independent of the angle ϕ. The solvability of (A.18) implies $e_z^{(1)} = 0$ and by (A.12), $e_x^{(1)} = \dfrac{\Lambda_0}{c}b_y^{(0)}$ where $\Lambda_0 = \dfrac{V_0}{\cos \alpha}$. This equation is then solved as

$$f_r^{(0)} = \mathcal{D}F_r^{(0)} \sin \phi \frac{b_y^{(1)}}{B_0} \tag{A.24}$$

with $\mathcal{D} = (\Lambda_0 - v_\parallel)\partial_{v_\perp} + v_\perp\partial_{v_\parallel}$. We also used the solvability condition of Eq. (A.19), that reads $\overline{f}_r^{(0)} = \langle f_r^{(0)}\rangle \equiv \frac{1}{2\pi} \int f_r^{(0)} d\phi = 0$.

It follows from the $\sin\phi$-dependence of $f_r^{(0)}$ that $\partial_\phi f_r^{(1)}$ only contains $\sin\phi$ and $\sin 2\phi$ Fourier modes.

The solvability condition of Eq. (A.20) reads

$$\frac{q_r}{m_r}e_z^{(2)}\partial_{v_\parallel}F_r^{(0)} - \frac{q_r}{2cm_r}\mathcal{D}^+\mathcal{D}F_r^{(0)}b_x^{(1)}\frac{b_y^{(1)}}{B_0} - \frac{q_r}{cm_r}\mathcal{D}^+\langle\sin\phi\partial_\phi f_r^{(1)}\rangle b_y^{(1)}$$
$$-v_\perp\sin\alpha\partial_\xi\langle\sin\phi\partial_\phi f_r^{(1)}\rangle + (v_\parallel\cos\alpha - V_0)\partial_\xi\overline{f}_r^{(1)} = 0, \tag{A.25}$$

where $\mathcal{D}^\pm = (\Lambda_0 - v_\parallel)(\partial_{v_\perp} \pm v_\perp^{-1}) + v_\perp\partial_{v_\parallel}$ and

$$\langle\sin\phi\partial_\phi f_r^{(1)}\rangle = -\frac{1}{2}\mathcal{D}F_r^{(0)}\frac{b_x^{(1)}}{B_0} + \frac{\cos\alpha}{2\Omega_r}(v_\parallel - \Lambda_0)\mathcal{D}F_r^{(0)}\partial_\xi\frac{b_y^{(1)}}{B_0}. \tag{A.26}$$

Assuming that the perturbations of the distribution function vanish at large ξ, we obtain

$$(v_\parallel - \Lambda_0)\overline{f}_r^{(1)} = (v_\parallel - \Lambda_0)R_r + S_r\partial_{v_\parallel}F_r^{(0)} \tag{A.27}$$

with

$$R_r = \frac{1}{2}\mathcal{D}^+\mathcal{D}F_r^{(0)}\frac{b_y^{(1)2}}{2B_0^2} - \frac{v_\perp}{2}\partial_{v_\perp}F_r^{(0)}A + \frac{\sin\alpha}{2\Omega_r}v_\perp\mathcal{D}F_r^{(0)}\partial_\xi\frac{b_y^{(1)}}{B_0} \tag{A.28}$$

$$S_r = \frac{q_r}{m_r}\varphi + \frac{1}{2}v_\perp^2 A, \tag{A.29}$$

where we have defined $A = \dfrac{b_z^{(1)}}{B_0} + \dfrac{b_y^{(1)2}}{2B_0^2}$ and $e_z^{(2)} = -\partial_z\varphi = \cos\alpha\partial_\xi\varphi$. As done in Ref.[16], the electric potential φ can be determined in terms of the magnetic perturbations A, using Eq. (A.4) that, to leading order, gives

$$\sin\alpha\frac{\Lambda_0}{c}\partial_\xi b_y^{(1)} = 4\pi\sum_r q_r n_r\int\overline{f}_r^{(1)}d^3v. \tag{A.30}$$

We do not use this approach here, but rather eliminate the electric potential using the expression of the density perturbations.[12]

Furthermore, one has[16]

$$\int\overline{f}_r^{(1)}dv_\parallel = \int R_r dv_\parallel + \mathcal{G}_r S_r \tag{A.31}$$

where we have defined the operator

$$\mathcal{G}_r = \mathrm{P}\int\frac{\partial_{v_\parallel}F_r^{(0)}}{v_\parallel - \lambda}dv_\parallel + \pi(\partial_{v_\parallel}F_r^{(0)})|_{v_\parallel=\lambda}\mathcal{H}_\xi, \tag{A.32}$$

$\mathcal{H}_\xi$ being the Hilbert transform with respect to the ξ variable.

The z-component of Eq. (A.3) (together the previously obtained condition $e_z^{(1)} = 0$), leads to the relation

$$\sin\alpha\partial_\xi b_y^{(1)} = \frac{4\pi}{c}\sum_r q_r n_r \int v_\parallel \overline{f}_r^{(1)} d^3v. \tag{A.33}$$

The x-component of Eq. (A.3) taken to leading order gives

$$-\cos\alpha\partial_\xi b_y^{(1)} = \frac{4\pi}{c}\sum_r q_r n_r \int v_\perp \cos\phi \widetilde{f}_r^{(1)} d^3v$$

$$= -\frac{4\pi}{c}\sum_r q_r n_r \int v_\perp \sin\phi\partial_\phi \widetilde{f}_r^{(1)} d^3v, \tag{A.34}$$

which, when using (A.19), provides the dispersion relation for oblique Alfvén waves, in the form

$$\Lambda_0^2 = v_A^2 + \frac{p_\perp^{(0)}}{\rho^{(0)}} - \frac{p_\parallel^{(0)}}{\rho^{(0)}}. \tag{A.35}$$

It involves the parallel and transverse pressures $p_\parallel^{(0)} = \sum_r p_{\parallel r}^{(0)}$ and $p_\perp^{(0)} = \sum_r p_{\perp r}^{(0)}$, together with the corresponding density $\rho^{(0)} = \sum_r \rho_r^{(0)}$. Here, $p_{\parallel r}^{(0)} = m_r n_r \int v_\parallel^2 F_r^{(0)} d^3v$, $p_{\perp r}^{(0)} = m_r n_r \int \frac{v_\perp^2}{2} F_r^{(0)} d^3v$ and $\rho_r^{(0)} = m_r n_r \int F_r^{(0)} d^3v$ denote the contributions of the various species to the above quantities. Furthermore, $v_A^2 = \frac{B_0^2}{4\pi\rho^{(0)}}$ is the Alfvén velocity.

Finally, the y-component of Eq. (A.3) gives

$$\partial_\xi(\cos\alpha b_x^{(1)} - \sin\alpha b_z^{(1)}) = \frac{4\pi}{c}\sum_r q_r n_r \int v_\perp \sin\phi \widetilde{f}_r^{(2)} d^3v, \tag{A.36}$$

that also rewrites

$$-\frac{1}{\sin\alpha}\partial_\xi \frac{b_z^{(1)}}{B_0} = \frac{4\pi}{cB_0}\sum_r q_r n_r \int v_\perp \cos\phi\partial_\phi \widetilde{f}_r^{(2)} d^3v. \tag{A.37}$$

It follows that

$$-\frac{1}{\sin\alpha}\partial_\xi \frac{b_z^{(1)}}{B_0} = -\frac{4\pi}{cB_0}\sum_r q_r n_r \int (\Lambda_0 - v_\parallel)\overline{f}_r^{(1)} d^3v \frac{b_y^{(1)}}{B_0}$$

$$+\frac{4\pi}{B_0^2}\sin\alpha\sum_r m_r n_r \int \frac{v_\perp^2}{2}\partial_\xi\left(\overline{f}_r^{(1)} - \frac{1}{2}\langle\sin 2\phi\partial_\phi \widetilde{f}_r^{(1)}\rangle\right)d^3v$$

$$-\frac{4\pi}{B_0}\cos\alpha\sum_r m_r n_r \int v_\perp(v_\parallel - \Lambda_0)\partial_\xi\langle\sin\phi\partial_\phi \widetilde{f}_r^{(1)}\rangle d^3v. \tag{A.38}$$

We are then led to compute

$$\langle\sin 2\phi\partial_\phi \widetilde{f}_r^{(1)}\rangle = \frac{1}{2}\mathcal{D}^-\mathcal{D}F_r^{(0)}\frac{b_y^{(1)2}}{2B_0^2} + \frac{\sin\alpha}{4\Omega_r}v_\perp \mathcal{D}F_r^{(0)}\partial_\xi \frac{b_y^{(1)}}{B_0}. \tag{A.39}$$

Using Eqs. (A.30) and (A.33) (with the condition $\Lambda_0 \ll c$) together with the relation

$$\sum_r m_r n_r \int \frac{v_\perp^2}{2} f_r^{(1)} d^3 v = p_\perp^{(1)} + \frac{\cos^2 \alpha}{\sin \alpha}\Big] \partial_\xi \frac{b_y^{(1)}}{B_0} \tag{A.40}$$

where, as previously, we neglect the mass m_e of the electrons compared to that m_p of the protons and defined $v_{\Delta r}^2 = \dfrac{p_{\perp r}^{(0)} - p_{\parallel r}^{(0)}}{\rho^{(0)}}$.

APPENDIX B: KINETIC FORM OF HYDRODYNAMIC QUANTITIES

1. Density fluctuations

The density fluctuations of the particles of r-species, defined to leading order as $\epsilon \rho_r^{(1)}$ with

$$\rho_r^{(1)} = m_r n_r \int \overline{f}_r^{(1)} d^3 v, \tag{B.1}$$

are given by

$$\rho_r^{(1)} = \mathcal{P}_r \varphi + (\rho_r^{(0)} + \mathcal{O}_r)A - \frac{\sin \alpha}{\Omega_r}\Lambda_0 \rho_r^{(0)} \partial_\xi \Big(\frac{b_y^{(1)}}{B_0}\Big), \tag{B.2}$$

where

$$\mathcal{P}_r = 2\pi q_r n_r \int_0^\infty \mathcal{G}_r d(\frac{v_\perp^2}{2}) \quad , \quad \mathcal{O}_r = 2\pi \sum_r m_r n_r \int_0^\infty \frac{v_\perp^2}{2}\mathcal{G}_r d(\frac{v_\perp^2}{2}). \tag{B.3}$$

The total density fluctuations are then given by $\rho^{(1)} = \sum_r \rho_r^{(1)}$.

2. Hydrodynamic velocities

The hydrodynamic velocity transverse to the local magnetic field is given by

$$U_\perp = \frac{\sum_r m_r n_r \int V_\perp f_r d^3 v}{\sum_r m_r n_r \int f_r d^3 v} \tag{B.4}$$

where $V_\perp = v - (v \cdot \widehat{b})\widehat{b}$ with $\widehat{b} = b/|b|$. One easily checks that $V_\perp \equiv \Big(V_{\perp x}, V_{\perp y}, V_{\perp z}\Big)$ with

$$V_{\perp x} = v_\perp \cos \phi - \epsilon v_\parallel \frac{b_x^{(1)}}{B_0} \tag{B.5}$$

$$V_{\perp y} = v_\perp \sin \phi - \epsilon^{1/2} v_\parallel \frac{b_y^{(1)}}{B_0} - \epsilon v_\perp \sin \phi \frac{b_y^{(1)2}}{B_0^2} \tag{B.6}$$

$$V_{\perp z} = -\epsilon^{1/2} v_\perp \sin \phi \frac{b_y^{(1)}}{B_0} + \epsilon(2v_\parallel \frac{b_y^{(1)2}}{2B_0^2} - v_\perp \cos \phi \frac{b_x^{(1)}}{B_0}) \tag{B.7}$$

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and thus

$$U_\perp \equiv \left(U_{\perp x}, U_{\perp y}, U_{\perp z} \right)$$
$$= \left(- \epsilon\Lambda_0 \frac{b_x^{(1)}}{B_0} - \epsilon\frac{\cos\alpha}{\Omega_p}(v_A^2 + v_{\Delta e}^2)\partial_\xi \left(\frac{b_y^{(1)}}{B_0}\right) \ , \ -\epsilon^{1/2}\Lambda_0\frac{b_y^{(1)}}{B_0} \ , \ \epsilon\Lambda_0\frac{b_y^{(1)2}}{B_0^2} \right) \tag{B.8}$$

where terms in m_e/m_p have been neglected.

The hydrodynamic velocity along the local magnetic field is

$$U_\parallel = \frac{\sum_r m_r n_r \int V_\parallel f_r d^3v}{\sum_r m_r n_r \int f_r d^3v} \tag{B.9}$$

with $V_\parallel = (v \cdot \widehat{b})\widehat{b}$. One gets $U_\parallel = \epsilon U_\parallel^{(1)} + \cdots$ with

$$U_\parallel^{(1)} = -\Lambda_0\frac{b_y^{(1)2}}{2B_0^2} + \frac{\Lambda_0}{\rho^{(0)}}\mathcal{P}\varphi + \frac{\Lambda_0}{\rho^{(0)}}\mathcal{O}A - \frac{\sin\alpha}{\Omega_p}v_{\Delta p}^2\partial_\xi\frac{b_y^{(1)}}{B_0}, \tag{B.10}$$

that also rewrites

$$U_\parallel^{(1)} = -\Lambda_0\frac{b_y^{(1)2}}{2B_0^2} + \Lambda_0\frac{\rho^{(1)}}{\rho^{(0)}} - \Lambda_0 A + \frac{\sin\alpha}{\Omega_p}(\Lambda_0^2 - v_{\Delta p}^2)\partial_\xi\frac{b_y^{(1)}}{B_0}. \tag{B.11}$$

By projecting $U_\perp + U_\parallel\widehat{b}$ on the three axes, we recover the hydrodynamic velocity components

$$u_x \approx -\epsilon\left(\Lambda_0\frac{b_x^{(1)}}{B_0} + (v_A^2 + v_{\Delta e}^2)\frac{\cos\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0}\right) \tag{B.12}$$

$$u_y \approx -\epsilon^{1/2}\Lambda_0\frac{b_y^{(1)}}{B_0} \tag{B.13}$$

$$u_z \approx \epsilon\left(\Lambda_0(\frac{\rho^{(1)}}{\rho^{(0)}} - \frac{b_z^{(1)}}{B_0}) + (v_A^2 + v_{\Delta e}^2)\frac{\sin\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0}\right). \tag{B.14}$$

3. Gyrotropic pressures

In the framework of a mono-fluid theory, the transverse and parallel components of the gyrotropic pressures are defined as

$$p_{\perp r} = m_r n_r \int \frac{1}{2}(V_\perp - U_\perp)^2 f_r d^3v \tag{B.15}$$

$$p_{\parallel r} = m_r n_r \int (V_\parallel - U_\parallel)^2 f_r d^3v. \tag{B.16}$$

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Defining the operators

$$\mathcal{M} = \sum_r \mathcal{M}_r = 2\pi \sum_r q_r n_r \int_0^\infty d(\frac{v_\perp^2}{2})\frac{v_\perp^2}{2}\mathcal{G}_r \qquad (\text{B.17})$$

$$\mathcal{N} = \sum_r \mathcal{N}_r = 2\pi \sum_r m_r n_r \int_0^\infty d(\frac{v_\perp^2}{2})\frac{v_\perp^4}{4}\mathcal{G}_r, \qquad (\text{B.18})$$

the leading pressure perturbations (or order ϵ) are given by

$$p_{\perp r}^{(1)} = m_r n_r \int \frac{v_\perp^2}{2}\overline{f}_r^{(1)}d^3v - (\Lambda_0^2\frac{\rho_r^{(0)}}{\rho^{(0)}} - v_{\Delta r}^2)\rho^{(0)}\frac{b_y^{(1)2}}{2B_0^2}$$

$$= \mathcal{M}_r\varphi + (2p_{\perp r}^{(0)} + \mathcal{N}_r)A - 2\frac{\sin\alpha}{\Omega_r}\Lambda_0 p_{\perp r}^{(0)}\partial_\xi\frac{b_y^{(1)}}{B_0} \qquad (\text{B.19})$$

and

$$p_{\parallel r}^{(1)} = m_r n_r \int \frac{v_\parallel^2}{2}\overline{f}_r^{(1)}d^3v - v_{\Delta r}^2\rho^{(0)}\frac{b_y^{(1)2}}{2B_0^2}$$

$$= (-q_r n_r + \Lambda_0^2\mathcal{P}_r)\varphi + (-\rho^{(0)}v_{\Delta r}^2 + \Lambda_0^2\mathcal{O}_r)A - \frac{\sin\alpha}{\Omega_r}\Lambda_0 p_{\parallel r}^{(0)}\partial_\xi\frac{b_y^{(1)}}{B_0}. \qquad (\text{B.20})$$

4.   Heat fluxes

a.   *Gyrotropic heat fluxes*

The gyrotropic components of the heat flux tensor

$$\mathbf{q}_r = \sum_r m_r n_r \int (v - U) \otimes (v - U) \otimes (v - U)f_r d^3v, \qquad (\text{B.21})$$

(U denoting the hydrodynamic velocity) read

$$q_{\perp r} = m_r n_r \int \frac{1}{2}(V_\perp - U_\perp)^2(V_\parallel - U_\parallel)f_r d^3v \qquad (\text{B.22})$$

$$q_{\parallel r} = m_r n_r \int (V_\parallel - U_\parallel)^3 f_r d^3v. \qquad (\text{B.23})$$

To leading order, one has $q_{\perp r} = \epsilon q_{\perp r}^{(1)} + \cdots$ and $q_{\parallel r} = \epsilon q_{\parallel r}^{(1)} + \cdots$ where

$$q_{\perp r}^{(1)} = -p_{\perp r}^{(0)}\left(U_\parallel^{(1)} + \Lambda_0\frac{b_y^{(1)2}}{2B_0^2}\right) + \Lambda_0\mathcal{M}_r\varphi + \Lambda_0\mathcal{N}_r A$$

$$+ \frac{\sin\alpha}{\Omega_r}m_r n_r \int (v_\parallel^2 v_\perp^2 - \frac{1}{4}v_\perp^4)F_r^{(0)}d^3v\partial_\xi\frac{b_y^{(1)}}{B_0}, \qquad (\text{B.24})$$

$$q_{\parallel r}^{(1)} = -3p_{\parallel r}^{(0)}\left(U_\parallel^{(1)} + \Lambda_0\frac{b_y^{(1)2}}{2B_0^2}\right) + \Lambda_0^3\rho_r^{(1)} - \Lambda_0(\Lambda_0^2\rho_r^{(0)} + p_{\perp r}^{(0)})A$$

$$- q_r n_r\Lambda_0\varphi + \frac{\sin\alpha}{\Omega_r}\left[\Lambda_0^4\rho_r^{(0)} + m_r n_r \int \left(v_\parallel^4 - \frac{3}{2}v_\perp^2 v_\parallel^2\right)d^3v\right]\partial_\xi\frac{b_y^{(1)}}{B_0}, \qquad (\text{B.25})$$

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with $U_\parallel^{(1)} + \Lambda_0 \dfrac{b_y^{(1)2}}{2B_0^2}$ given by Eq. (B.11).

b. Heat flux contributions to the gyrotropic pressures

The longitudinal and transverse pressures (relatively to the local magnetic field) involve $\mathrm{tr}\nabla \cdot \mathbf{q}_r$ and $\widehat{b} \cdot \nabla \cdot \mathbf{q}_r \cdot \widehat{b}$. The heat flux components being also of order ϵ, the distortion of the magnetic field lines can be neglected to leading order. We are thus led to write

$$\mathrm{tr}\ \nabla \cdot \mathbf{q}_r \approx \partial_x(q_{r\,111} + q_{r\,221} + q_{r\,331}) + \partial_z(q_{\perp r} + q_{\parallel r}) \tag{B.26}$$

and

$$\widehat{b} \cdot (\nabla \cdot \mathbf{q}_r) \cdot \widehat{b} \approx \partial_x q_{r\,331} + \partial_z q_{\parallel r}, \tag{B.27}$$

where the $q_{r\,ijk}$'s hold for the components in the local frame of the heat flux associated with the particles of species r. One has

$$q_{r\,331} \equiv \int (V_\parallel - U_\parallel)^2(V_{\perp x} - U_{\perp x}) f_r d^3 v = \epsilon q_{r\,331}^{(1)} + \cdots \tag{B.28}$$

with

$$q_{r\,331}^{(1)} = \left\{ \frac{\cos\alpha}{\Omega_p} p_{\parallel r}^{(0)}\left[v_A^2 + v_{\Delta e}^2 - \frac{\Omega_p}{\Omega_r}(v_A^2 + v_{\Delta p}^2 + v_{\Delta e}^2) \right] - \frac{\cos\alpha}{\Omega_r} \int \left(v_\parallel^4 - \frac{3}{2}v_\perp^2 v_\parallel^2\right) F_r^{(0)} d^3 v \right\} \partial_\xi \frac{b_y^{(0)}}{B_0}. \tag{B.29}$$

Similarly, to leading order

$$q_{r\,111} + q_{r\,221} = \int (V_\perp - U_\perp)^2(V_{\perp x} - U_{\perp x}) f_r d^3 v = \epsilon(q_{111,r}^{(1)} + q_{221,r}^{(1)}) + \cdots \tag{B.30}$$

with

$$q_{r\,111}^{(1)} + q_{r\,221}^{(1)} = \left\{ -\frac{\cos\alpha}{\Omega_r}\left[4p_{\perp r}^{(0)}\Lambda_0^2 + 2m_r n_r \int \left(v_\parallel^2 v_\perp^2 - \frac{1}{4}v_\perp^4\right) F_r^{(0)} d^3 v \right] + 4\frac{\cos\alpha}{\Omega_p} p_{\perp r}^{(0)}(v_A^2 + v_{\Delta e}^2) \right\} \partial_\xi \frac{b_y^{(0)}}{B_0}. \tag{B.31}$$

c. Heat flux contribution to the non-gyrotropic pressures

Neglecting the magnetic field line distortions that are irrelevant at the considered order, we write the non-gyrotropic contribution to $\nabla \cdot \mathbf{q}_p$ in the form

$$\overline{\overline{\nabla \cdot \mathbf{q}_p}} = \partial_k \begin{pmatrix} \frac{1}{2}(q_{p\,11k} - q_{p\,22k}) & q_{p\,12k} & q_{p\,13k} \\[1ex] q_{p\,21k} & -\frac{1}{2}(q_{p\,11k} - q_{p\,22k}) & q_{p\,23k} \\[1ex] q_{p\,31k} & q_{p\,32k} & 0 \end{pmatrix} \tag{B.32}$$

where we concentrate on the proton contribution. To leading order $q_{pijk} = \int (v_i - u_i)(v_j - u_j)(v_k - u_k) f_p d^3 v \approx \epsilon q^{(1)}_{pijk}$. Since we assume no dependency in the y-variable, we are led to compute

$$q^{(1)}_{p\,111} = -3 p^{(0)}_{\perp p} u^{(1)}_x + m_p n_p \int v_\perp^3 \cos^3 \phi f^{(1)}_p d^3 v \tag{B.33}$$

$$q^{(1)}_{p\,221} = -p^{(0)}_{\perp p} u^{(1)}_x + m_p n_p \int v_\perp^3 \sin^2 \phi \cos \phi f^{(1)}_p d^3 v \tag{B.34}$$

$$q^{(1)}_{p\,112} = m_p n_p \int v_\perp^3 \cos^2 \phi \sin \phi f^{(1)}_p d^3 v \tag{B.35}$$

$$q^{(1)}_{p\,113} = -p^{(0)}_{\perp p} u^{(1)}_z + m_p n_p \int v_\perp^2 v_\| \cos^2 \phi f^{(1)}_p d^3 v \tag{B.36}$$

$$q^{(1)}_{p\,223} = -p^{(0)}_{\perp p} u^{(1)}_z + 2(p^{(0)}_{\perp p} - p^{(0)}_{\| p}) u_y \frac{b^{(1)}_y}{B_0} + m_p n_p \int v_\perp^2 v_\| \sin^2 \phi f^{(1)}_p d^3 v \tag{B.37}$$

$$q^{(1)}_{p\,123} = m_p n_p \int v_\perp^2 v_\| \sin \phi \cos \phi f^{(1)}_p d^3 v \tag{B.38}$$

$$q^{(1)}_{p\,133} = -p^{(0)}_{\| p} u^{(1)}_x + m_p n_p \int v_\perp v_\|^2 \cos \phi f^{(1)}_p d^3 v \tag{B.39}$$

$$q^{(1)}_{p\,233} = m_p n_p \int v_\perp v_\|^2 \sin \phi f^{(1)}_p d^3 v. \tag{B.40}$$

Since $f^{(1)}_p$ only projects on 1, $\cos \phi$ and $\cos 2\phi$, one has

$$q^{(1)}_{p\,112} = q^{(1)}_{p\,123} = q^{(1)}_{p\,233} = 0, \tag{B.41}$$

and the only integrals to be computed read

$$m_p n_p \int v_\perp^3 \sin \phi \partial_\phi f^{(1)}_p d^3 v = 4 \Lambda_0 p^{(0)}_{\perp p} \frac{b^{(1)}_x}{B_0} + \left(4 \Lambda_0^2 p^{(0)}_{\perp p} \right.$$
$$\left. + 4 m_p n_p \int (v_\|^2 v_\perp^2 - \frac{v_\perp^4}{4}) F^{(0)}_p d^3 v \right) \frac{\cos \alpha}{\Omega_p} \partial_\xi \frac{b^{(1)}_y}{B_0} \tag{B.42}$$

$$m_p n_p \int v_\perp v_\|^2 \sin \phi \partial_\phi f^{(1)}_p d^3 v = \Lambda_0 p^{(0)}_{\| p} \frac{b^{(1)}_x}{B_0}$$
$$+ \left(\Lambda_0^2 p^{(0)}_{\| p} + m_p n_p \int (v_\|^4 - \frac{3}{2} v_\perp^2 v_\|^2) F^{(0)}_p d^3 v \right) \frac{\cos \alpha}{\Omega_p} \partial_\xi \frac{b^{(1)}_y}{B_0} \tag{B.43}$$

$$m_p n_p \int v_\perp^2 v_\| \sin 2\phi \partial_\phi f^{(1)}_p d^3 v = -8 \Lambda_0 (p^{(0)}_{\perp p} - p^{(0)}_{\| p}) \frac{b^{(1)2}_y}{2 B_0^2}$$
$$+ \int (v_\|^2 v_\perp^2 - \frac{v_\perp^4}{4}) F^{(0)}_p d^3 v \frac{\sin \alpha}{\Omega_p} \partial_\xi \frac{b^{(1)}_y}{B_0}. \tag{B.44}$$

The fourth-order velocity moments are explicited in Appendix C where bi-Maxwellian distribution functions are assumed for the equilibrium state.

APPENDIX C: EQUILIBRIUM BI-MAXELLIAN DISTRIBUTION

It is possible to simplify the above general expressions for the hydrodynamic moments by assuming that the plasma contains electrons and only one species of ions (with $Z = 1$), with bi-Maxwellian equilibrium distribution functions

$$F_r^{(0)} = \frac{1}{(2\pi)^{3/2}} \frac{m_r^{3/2}}{T_{\perp r}^{(0)} T_{\parallel r}^{(0)1/2}} \exp\left\{ - \left(\frac{m_r}{2T_{\parallel r}^{(0)}} v_{\parallel}^2 + \frac{m_r}{2T_{\perp r}^{(0)}} v_{\perp}^2 \right) \right\}. \tag{C.1}$$

Using the quasi-neutrality condition that prescribes $n_r = n^{(0)}$ and $\rho_r^{(1)} = m_r n^{(1)}$, one obtains

$$\mathcal{M}_r = -n^{(0)} q_r \frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}} \mathcal{W}_r \ , \ \mathcal{N}_r = -2n^{(0)} \frac{T_{\perp r}^{(0)2}}{T_{\parallel r}^{(0)}} \mathcal{W}_r \ , \tag{C.2}$$

$$\mathcal{O}_r = -n^{(0)} m_r \frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}} \mathcal{W}_r \ , \ \mathcal{P}_r = -n^{(0)} m_r q_r \frac{1}{T_{\parallel r}^{(0)}} \mathcal{W}_r \tag{C.3}$$

where, normalizing the propagation velocity of the wave by the thermal velocity $v_{th,r} = \sqrt{T_{\parallel r}^{(0)}/m_r}$ in the form $c_r = \Lambda_0/v_{th,r}$, one writes

$$\mathcal{W}_r \equiv \mathcal{W}(c_r) = \frac{1}{\sqrt{2\pi}} \mathrm{P} \int \frac{\zeta e^{-\zeta^2/2}}{\zeta - c_r} d\zeta + \sqrt{\frac{\pi}{2}} c_r e^{-c_r^2/2} \mathcal{H}_\xi, \tag{C.4}$$

or[42]

$$\mathcal{W}(c_r) = 1 - c_r e^{-\frac{c_r^2}{2}} \int_0^{c_r} e^{\frac{\zeta^2}{2}} d\zeta + \sqrt{\frac{\pi}{2}} c_r e^{-c_r^2/2} \mathcal{H}_\xi. \tag{C.5}$$

This function is related to the plasma response function $\mathcal{R}$ used by SHD by $\mathcal{W}(X) = \mathcal{R}(X/\sqrt{2})$.

This leads to express

$$\frac{p_{\parallel r}^{(1)}}{p_{\parallel r}^{(0)}} = \left(1 - \frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}} - \frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}} c_r^2 \mathcal{W}_r \right) A + (c_r^2 + \mathcal{W}_r^{-1}) \left(\frac{n^{(1)}}{n^{(0)}} - (1 - \frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}} \mathcal{W}_r) A \right)$$

$$+ (c_r^2 - 1 + \mathcal{W}_r^{-1}) \frac{\sin \alpha}{\Omega_r} \Lambda_0 \partial_\xi \frac{b_y^{(1)}}{B_0} \tag{C.6}$$

and

$$\frac{T_{\parallel r}^{(1)}}{T_{\parallel r}^{(0)}} \equiv \frac{p_{\parallel r}^{(1)}}{p_r^{(0)}} - \frac{n^{(1)}}{n^{(0)}}$$

$$= (c_r^2 - 1 + \mathcal{W}_r^{-1}) \left(\frac{n^{(1)}}{n^{(0)}} - A + \frac{\sin \alpha}{\Omega_r} \Lambda_0 \partial_\xi \frac{b_y^{(1)}}{B_0} \right). \tag{C.7}$$

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Similarly,

$$
\begin{aligned}
\frac{T_{\perp r}^{(1)}}{T_{\perp r}^{(0)}} &\equiv \frac{p_{\perp r}^{(1)}}{p_{\perp r}^{(0)}} - \frac{n^{(1)}}{n^{(0)}} \\
&= \left(1 - \frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}}\mathcal{W}_r\right)A - 3\frac{\sin\alpha}{\Omega_r}c_r v_{thr}\partial_\xi \frac{b_y^{(1)}}{B_0}.
\end{aligned}
\tag{C.8}
$$

When considering the heat flux components, we have to evaluate the integrals

$$
m_r n_r \int (v_\parallel^4 - \frac{3}{2}v_\parallel^2 v_\perp^2)F_r^{(0)}d^3v = -3v_{\Delta r}^2 p_{\parallel r}^{(0)}
\tag{C.9}
$$

and

$$
m_r n_r \int (v_\parallel^2 v_\perp^2) - \frac{1}{4}v_\perp^4)F_r^{(0)}d^3v = -2v_{\Delta r}^2 p_{\perp r}^{(0)}.
\tag{C.10}
$$

We get

$$
\frac{q_{\parallel r}^{(1)}}{v_{th,r}p_{\parallel r}^{(0)}} = c_r\frac{c_r^2 - 3 + W_r^{-1}}{c_r^2 - 1 + W_r^{-1}}\frac{T_{\parallel r}^{(1)}}{T_{\parallel r}^{(0)}} + 3\left[\left(c_r^2 - \frac{v_{\Delta r}^2}{v_{thr}^2}\right)\left(\frac{\Omega_p}{\Omega_r} - 1\right) + \frac{v_{\Delta p}^2 - v_{\Delta r}^2}{v_{th,r}^2}\right]\frac{\sin\alpha}{\Omega_p}v_{th,r}\partial_\xi\frac{b_y^{(1)}}{B_0}
\tag{C.11}
$$

and

$$
\frac{q_{\perp r}^{(1)}}{v_{th,r}p_{\perp r}^{(0)}} = -\frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}}c_r W_r A + \left[-c_r^2\left(\frac{\Omega_p}{\Omega_r} + 1\right) + \frac{v_{\Delta p}^2}{v_{th,r}^2} - 2\frac{v_{\Delta r}^2}{v_{th,r}^2}\frac{\Omega_p}{\Omega_r}\right]\frac{\sin\alpha}{\Omega_p}v_{th,r}\partial_\xi\frac{b_y^{(1)}}{B_0},
\tag{C.12}
$$

that also rewrites

$$
\begin{aligned}
\frac{q_{\perp r}^{(1)}}{v_{th,r}p_{\perp r}^{(0)}} &= -\frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}}\frac{c_r W_r}{1 - \frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}}W_r}\left[\frac{T_{\perp r}^{(1)}}{T_{\perp r}^{(0)}} + 3c_r\frac{\sin\alpha}{\Omega_r}v_{th,r}\partial_\xi\frac{b_y^{(1)}}{B_0}\right] \\
&\quad + \left[-c_r^2\left(\frac{\Omega_p}{\Omega_r} + 1\right) + \frac{v_{\Delta p}^2}{v_{th,r}^2} - 2\frac{v_{\Delta r}^2}{v_{th,r}^2}\frac{\Omega_p}{\Omega_r}\right]\frac{\sin\alpha}{\Omega_p}v_{th,r}\partial_\xi\frac{b_y^{(1)}}{B_0}.
\end{aligned}
\tag{C.13}
$$

Note that for $\alpha = 0$, the parallel and transverse energy fluxes computed in[12,16] in the case of parallel Alfvén waves are recovered.

Furthermore, the non-zero coefficients entering the non-gyrotropic proton heat flux components considered in Appendix B, become

$$
q_{p\,111}^{(1)} - q_{p\,221}^{(1)} = 0
\tag{C.14}
$$

$$
q_{p\,113}^{(1)} - q_{p\,223}^{(1)} = p_{\perp p}^{(0)}v_{\Delta p}^2\frac{\sin\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0}
\tag{C.15}
$$

$$
q_{p\,133}^{(1)} = 2p_{\parallel p}^{(0)}v_{\Delta p}^2\frac{\cos\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0},
\tag{C.16}
$$

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together with

$$q_{p\,113}^{(1)} = -p_{\perp p}^{(0)} u_z^{(1)} + \Lambda_0 p_{\perp p}^{(1)} - 2\Lambda_0 p_{\perp p}^{(0)} A + \Lambda_0 p_{\perp p}^{(0)} \frac{b_y^{(1)2}}{2B_0^2}$$

$$+ (2\Lambda_0^2 - \frac{3}{2} v_{\Delta p}^2) p_{\perp p}^{(0)} \frac{\sin\alpha}{\Omega_p} \partial_\xi \frac{b_y^{(1)}}{B_0}$$

$$= \Lambda_0 p_{\perp p}^{(0)} \left(\frac{p_{\perp p}^{(1)}}{p_{\perp p}^{(0)}} - \frac{\rho_p^{(1)}}{\rho_p^{(0)}} - A \right) + (\Lambda_0^2 - \frac{v_{\Delta p}^2}{2}) p_{\perp p}^{(0)} \frac{\sin\alpha}{\Omega_p} \partial_\xi \frac{b_y^{(1)}}{B_0}$$

$$= q_{\perp p}^{(1)} + p_{\perp p}^{(0)} v_{\Delta p}^2 \frac{\sin\alpha}{2\Omega_p} \partial_\xi \frac{b_y^{(1)}}{B_0} \qquad (C.17)$$

where we have used Eqs. (B.24), (B.11) and (B.19).

Equation (B.32) then reads

$$\overline{\overline{\nabla \cdot \mathbf{q}_p}} = \begin{pmatrix} \lambda_p & 0 & \mu_p \\ 0 & -\lambda_p & 0 \\ \mu_p & 0 & 0 \end{pmatrix} \qquad (C.18)$$

with

$$\lambda_p = p_{\perp p}^{(0)} \frac{v_{\Delta p}^2}{2\Omega_p} \partial_{xz} \frac{b_y^{(1)}}{B_0} \qquad (C.19)$$

$$\mu_p = \partial_x q_{\perp p} + p_{\perp p}^{(0)} \frac{v_{\Delta p}^2}{2\Omega_p} \partial_{xx} \frac{b_y^{(1)}}{B_0} + 2 p_{\| p}^{(0)} \frac{v_{\Delta p}^2}{\Omega_p} \partial_{zz} \frac{b_y^{(1)}}{B_0}. \qquad (C.20)$$

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