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CONVERGENCE OF A SEMI-LAGRANGIAN SCHEME FOR THE ONE-DIMENSIONAL VLASOV–POISSON SYSTEM*

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Abstract. A semi-Lagrangian scheme is proposed for solving the periodic one-dimensional Vlasov–Poisson system in phase space on unstructured meshes. The distribution function $f(t, x, v)$ and the electric field $E(t, x)$ are shown to converge to the exact solution values in the L^∞ norm. The rate of convergence is in $O(h^{4/3})$.

Key words. Vlasov–Poisson system, semi-Lagrangian methods, convergence analysis

AMS subject classifications. 65M12, 82D10

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1. Introduction. The numerical resolution of the Vlasov equation is usually performed by Lagrangian methods like particles-in-cell methods (PIC), which consist of approximating the plasma by a finite number of macroparticles. The trajectories of these particles are computed from the characteristic curves given by the Vlasov equation, whereas self-consistent fields are computed by gathering the charge and current densities of the particles on a mesh of the physical space (see Birdsall and Langdon [10] for more details). Although this method allows us to obtain satisfying results with a small number of particles, it is well known that the numerical noise inherent to the particle method becomes too large to allow a precise description of the tail of the distribution function, which plays an important role in charged particle beams. To remedy this problem, Eulerian methods have been proposed which consist of discretizing the Vlasov equation on a mesh of phase space. For example, finite volume schemes, which are known to be robust and computationally cheap, have been implemented by Boris and Book [11], Cheng and Knorr [13], and more recently Mineau [32], Fijalkow [19], and Filbet, Sonnendrücker, and Bertrand [21]. Nevertheless, finite volume schemes are low order, too dissipative, and restricted by a CFL condition.

Other kinds of Eulerian method are the semi-Lagrangian methods which, in some particular cases, can be regarded as local versions of characteristic Galerkin methods [3, 4], which have been used in convection-diffusion problems [17, 35, 25]. Semi-Lagrangian methods were introduced at the beginning of the 1980s for the time-advection of various atmospheric and fluid dynamics models [43, 42, 37], which can be formulated as abstract Liouville systems (ALS). Semi-Lagrangian advection attempts to combine the advantages of both Eulerian and Lagrangian advection schemes while ameliorating their drawbacks. Eulerian advection schemes have good resolution properties, but CFL condition number, which is a necessary condition for achieving numerical stability, often leads to overly restrictive time steps. On the other hand, Lagrangian advection schemes allow one to use larger time steps, but, at later times, Lagrangian distortion (an initial regularly spaced set of particles will generally become highly irregularly spaced over long times) implies that important features of the flow

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may not be well described. A semi-Lagrangian method uses a regular Cartesian mesh and different sets of particles. At each time step the set of particles is chosen such that they arrive exactly at the points of the mesh at the end of the time step, and is advected by the characteristic curves of the ALS. More precisely, the method consists of directly computing the distribution function of the ALS on a fixed Cartesian grid of phase space, by integrating (or following) the characteristic curves backward (from the end of the characteristic, which is a point of the fixed mesh, to the beginning of characteristic, during a time step) at each time step and interpolating the value at the base of the characteristics. In recent applications of semi-Lagrangian methods to lower-dimensional relativistic Vlasov–Maxwell (RVM) calculations [1, 2, 40], cubic splines are used for the interpolation scheme, linear interpolation being too dissipative. Semi-Lagrangian methods have been efficiently implemented using parallel computers [41] and give considerable promise for displaying the detailed structure of distribution functions in weak density regions.

The author extends semi-Lagrangian schemes on unstructured meshes with a different kind of high order local interpolation operator and with the possibility of having a positive and conservative method by introducing a linear combination of low order solutions and high order solutions tempered by a limiter coefficient (cf. [9]). Here we present the convergence of the method for the simplest interpolation operator, that is, the Lagrange first order interpolation operator. The scheme preserves positivity because the basis functions associated with the Lagrange first order interpolation operator are always positive. Additionally, the scheme is not limited by a CFL condition. More complicated interpolation on a triangle, which involves knowledge of the gradient of the distribution function, has been implemented successfully (cf. [9]), but it seems to be a challenge to show the convergence of these methods because we advect not only the distribution function f but also its gradients. A first result on the convergence analysis of semi-Lagrangian methods with propagation of gradients is stated in [8].

Let us note that a first work on convergence of one-dimensional particle methods is [33], where Neunzert and Wick consider nonuniform initial loadings of particles asymptotically distributed with respect to initial data. Cottet and Raviart [16] present a mathematical analysis of the particle method for solving the one-dimensional Vlasov–Poisson system, where uniform initial loadings of particles are considered. A number of additional authors have studied the convergence of particle methods for the multidimensional Vlasov–Poisson system [22, 45, 46, 49]. They have also proved convergence results on random and deterministic particle methods for the Vlasov–Poisson–Fokker–Planck kinetic equations [26, 27]. Finally, Glassey and Schaeffer have done the convergence analysis of a particle method for the RVM system [24]. Schaeffer [39] has also proved the convergence of a finite difference scheme for the one-dimensional Vlasov–Poisson–Fokker–Planck system, and Filbet [20] has shown the convergence of a finite volume scheme for the one-dimensional Vlasov–Poisson system.

Although a number of papers present satisfactory numerical results using semi-Lagrangian methods [43, 13, 40, 1, 2, 18, 9], few rigorous mathematical results on convergence analysis of semi-Lagrangian methods have been stated. Although interesting a priori estimates have been pointed out (cf. [4, 5, 18]), a lot of work still remains to give complete and rigorous results in more general situations. The more difficult step in the convergence analysis of semi-Lagrangian methods is obtaining a stability result for the interpolation operators. If stability results in the L^∞ norm seem inaccessible for high order interpolation operators because of the Runge phenomena (artificial oscillations, whose amplitude increases with the degree of the polynomial in the case of Lagrange interpolation, appear at the edges of finite elements), a more appropriate mathematical framework is L^2 stability. If Fourier analysis tools as Fourier series are useful for proving L^2 stability in the case of grids, convenient mathematical tools are lacking for unstructured meshes such as triangulation and have to be developed in the future. Nevertheless new results on the convergence analysis of classes of high order schemes can be found in [7, 8, 6].

This paper is organized as follows. In the first part we present the continuous problem. In the second part we expose the discrete problem and the numerical scheme to solve it. Then we study the convergence of our numerical scheme. In the last section we give refined convergence results.

2. The continuous problem. We consider a noncollisional plasma of charged particles (electrons and ions) in one dimension. We take into account the electrostatic forces and neglect the magnetic effects. Due to the great inertia of the ions compared to the electrons, we assume that the ions form a neutralizing uniform background.

Denoting by $f(t,x,v) \geq 0$ the distribution function of electrons in phase space (with mass normalized to one, the charge to plus one), and by $E(t,x)$ the self-consistent electric field, the adimensional Vlasov–Poisson system reads

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + E(t,x)\frac{\partial f}{\partial v} = 0, \tag{2.1}$$

$$\frac{dE}{dx}(t,x) = \rho(t,x) = \int_{-\infty}^{+\infty} f(t,x,v)dv - 1. \tag{2.2}$$

We consider a periodic plasma of period L. Hence in (2.1) and (2.2) we have $x \in [0,L]$, $v \in \mathbb{R}$, $t \geq 0$, and the functions f and E satisfy the periodic boundary conditions

$$f(t,0,v) = f(t,L,v), \quad v \in \mathbb{R}, \quad \mathrm{t} \geq 0, \tag{2.3}$$

and

$$E(t,0) = E(t,L) \Longleftrightarrow \frac{1}{L}\int_0^L \int_{-\infty}^{+\infty} f(t,x,v)dvdx = 1, \quad t \geq 0, \tag{2.4}$$

which means that the plasma is globally neutral. In order to have a well-posed problem, we add to (2.1)–(2.4) a zero-mean electrostatic condition,

$$\int_0^L E(t,x)dx = 0, \quad t \geq 0, \tag{2.5}$$

and an initial condition,

$$f(0,x,v) = f_0(x,v), \quad x \in [0,L], \; v \in \mathbb{R}. \tag{2.6}$$

If we introduce the electrostatic potential $\phi = \phi(t,x)$ such that

$$E(t,x) = -\frac{\partial \phi}{\partial x}(t,x),$$

and if we denote by $G = G(x,y)$ the Green function associated with our problem—that is to say, for $y \in]0,L[$, $G(.,y)$ is the solution of

$$-\frac{\partial^2 G}{\partial x^2}(x,y) = \delta(x-y), \quad x \in [0,L], \quad G(0,y) = G(L,y),$$

where δ is the Dirac distribution—then $G(x,y)$ and $K(x,y) = -\partial_x G(x,y)$ are given by

$$G(x,y) = \begin{cases} x\left(1 - \dfrac{y}{L}\right), & 0 \le x \le y, \\ y\left(1 - \dfrac{x}{L}\right), & y \le x \le L, \end{cases} \qquad K(x,y) = \begin{cases} \left(\dfrac{y}{L} - 1\right), & 0 \le x < y, \\ \dfrac{y}{L}, & y < x \le L. \end{cases}$$

Therefore ϕ is given by

$$\phi(t,x) = \int_0^L G(x,y)\left(\int_{-\infty}^{+\infty} f(t,y,v)dv - 1\right)dy,$$

and E can be rewritten as

$$E(t,x) = \int_0^L K(x,y)\left(\int_{-\infty}^{+\infty} f(t,y,v)dv - 1\right)dy. \tag{2.7}$$

In addition, assuming that the electric field E is smooth enough, we can solve (2.1), (2.3), and (2.6) in the classical sense as follows. For the existence, uniqueness, and regularity of the solutions of the following differential system we refer the reader to [12] and [36].

We consider the first order differential system

$$\begin{aligned} \frac{dX}{dt}(t;s,x,v) &= V(t;s,x,v), \\ \frac{dV}{dt}(t;s,x,v) &= E(t, X(t;s,x,v)) \end{aligned} \tag{2.8}$$

and denote by $t \to (X(t;s,x,v), V(t;s,x,v))$ the characteristic curves, which are the solution of (2.8) with the initial conditions

$$X(s;s,x,v) = x, \qquad V(s;s,x,v) = v. \tag{2.9}$$

Then the solution of problem (2.1), (2.6) is given by

$$f(t,x,v) = f_0(X(0;t,x,v), V(0;t,x,v)), \qquad x, v \in \mathbb{R}, \ \ t \ge 0. \tag{2.10}$$

We note that the periodicity in x of $f_0(x,v)$ and $E(t,x)$ implies the periodicity in x of $f(t,x,v)$. Moreover, as

$$\left|\frac{\partial(X,V)}{\partial(x,v)}\right| = 1,$$

we get

$$\frac{1}{L}\int_0^L \int_{-\infty}^{+\infty} f(t,x,v)dvdx = \frac{1}{L}\int_0^L \int_{-\infty}^{+\infty} f_0(x,v)dvdx = 1.$$

Therefore, according to the previous considerations, an equivalent form of the Vlasov–Poisson periodic problem is to find a pair (f,E), smooth enough, periodic with respect to x, with period L, and solving (2.7), (2.8), (2.9), and (2.10).

2.1. Definitions and notation. We now introduce basic notation. If $\mathbb{N}$ denotes the set of nonnegative integers, a multi-index α is an n-tuple of nonnegative integers $\alpha := (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N}$, $i = 1, \ldots, n$. We have the following definitions:

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$

$$D^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}.$$

Let Ω be a domain in $\mathbb{R}^n$. For any nonnegative integer m let $\mathscr{C}^m(\Omega)$ be the vector space consisting of all functions ϕ that, together with all their partial derivatives $D^\alpha \phi$ of orders $|\alpha| \leq m$, are continuous on Ω.

We define the vector space $\mathscr{C}_b^m(\Omega)$ of all functions $\phi \in \mathscr{C}^m(\Omega)$ for which $D^\alpha \phi$ is bounded and uniformly continuous on Ω for $0 \leq |\alpha| \leq m$. $\mathscr{C}_b^m(\Omega)$ is a Banach space with the norm given by

$$||\phi||_{\mathscr{C}_b^m(\Omega)} = \max_{0 \leq |\alpha| \leq m} \sup_{z \in \Omega} |D^\alpha \phi(z)|.$$

We define $\mathscr{C}_c^m(\Omega)$ as the subspace of $\mathscr{C}_b^m(\Omega)$ consisting of those functions ϕ for which, for $0 \leq |\alpha| \leq m$, $D^\alpha \phi$ has compact support in Ω.

If $0 < \lambda \leq 1$, we define $\mathscr{C}^{m,\lambda}(\Omega)$ to be the subspace of $\mathscr{C}_b^m(\Omega)$ consisting of those functions ϕ for which, for $0 \leq |\alpha| \leq m$, $D^\alpha \phi$ satisfies in Ω a Hölder condition of exponent λ; that is, there exists a constant K such that

$$|D^\alpha \phi(x) - D^\alpha \phi(y)| \leq K|x - y|^\lambda, \quad x, y \in \Omega.$$

$\mathscr{C}^{m,\lambda}(\Omega)$ is a Banach space with norm given by

$$||\phi||_{\mathscr{C}^{m,\lambda}(\Omega)} = ||\phi||_{\mathscr{C}_b^m(\Omega)} + \max_{0 \leq |\alpha| \leq m} \sup_{\substack{x,\, y \,\in\, \Omega \\ x \neq y}} \frac{|D^\alpha \phi(x) - D^\alpha \phi(y)|}{|x - y|^\lambda}.$$

For all $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ we let

$$\mathrm{Lip}(\phi) = \sup_{\substack{x,\, y \,\in\, \Omega \\ x \neq y}} \frac{|\phi(x) - \phi(y)|}{|x - y|}.$$

Furthermore,

$$\mathrm{Lip}(\Omega) = \{\phi : \mathbb{R}^n \longrightarrow \mathbb{R} \mid \ \mathrm{Lip}(\phi) < \infty\}$$

is a Banach space with the norm given by

$$||\phi||_{\mathrm{Lip}(\Omega)} = ||\phi||_{\mathscr{C}^{0,1}(\Omega)}.$$

We define $\mathscr{C}_{b,per_{x_i}}^m(\Omega_{x_i} \times \Omega_{n-1})$ as the subspace of $\mathscr{C}_b^m(\Omega)$ consisting of those functions ϕ which are periodic with respect to the variable x_i and bounded with respect to other variables. We also define $\mathscr{C}_{c,per_{x_i}}^m(\Omega_{x_i} \times \Omega_{n-1})$ as the subspace of $\mathscr{C}_c^m(\Omega)$ consisting of those functions ϕ which are periodic with respect to the variable x_i and compactly supported with respect to the other variables.

We denote by $L^p(\Omega)$, $1 \le p \le \infty$, the space of all equivalence classes of real-valued Lebesgue-measurable functions. $L^p(\Omega)$ is a Banach space with the norm given by

$$||\phi||_{L^p(\Omega)} = \left\{ \int_\Omega |\phi|^p d\Omega \right\}^{1/p}, \quad 1 \le p < \infty,$$
$$||\phi||_{L^\infty(\Omega)} = \text{ess} \sup_{z\in\Omega} |\phi(z)|.$$

We define $W^{m,p}(\Omega)$ to be the Sobolev space consisting of all functions ϕ which, together with all their partial derivatives $D^\alpha \phi$ taken in the sense of distribution of orders $|\alpha| \le m$, belong to the $L^p(\Omega)$ space. If we define the seminorm as

$$|\phi|_{W^{k,p}(\Omega)} = \left\{ \sum_{|\alpha|=k} |D^\alpha \phi|^p_{L^p(\Omega)} \right\}^{1/p}, \quad 1 \le p < \infty,$$
$$|\phi|_{W^{k,\infty}(\Omega)} = \max_{|\alpha|=m} \ \text{ess} \sup_{z\in\Omega} |D^\alpha \phi(z)|,$$

then we provide $W^{m,p}(\Omega)$ with the norm

$$||\phi||_{W^{m,p}(\Omega)} = \left\{ \sum_{k=0}^{m} |\phi|^p_{W^{k,p}(\Omega)} \right\}^{1/p}, \quad 1 \le p < \infty,$$
$$||\phi||_{W^{m,\infty}(\Omega)} = \max_{0\le k\le m} |\phi|_{W^{k,\infty}(\Omega)}.$$

Let X be a Banach space with norm $||\cdot||_X$. We denote by $\mathscr{C}^m(0,T;X)$, $0 < T < +\infty$, the space of m-times continuously differentiable functions from $(0,T)$ into X, and by $L^p(0,T;X)$ the space of all strongly measurable functions $\phi : t \longrightarrow \phi(t)$ from $(0,T)$ into X. The following norms are defined:

$$||\phi||_{\mathscr{C}(0,T;X)} = \sup_{t\in[0,T]} ||\phi(t)||_X,$$

$$||\phi||_{\mathscr{C}^m(0,T;X)} = \sum_{k=0}^{m} \left\| \frac{d^k \phi}{dt^k} \right\|_{\mathscr{C}(0,T;X)},$$

$$||\phi||_{L^p(0,T;X)} = \left\{ \int_0^T ||\phi(t)||^p_X dt \right\}^{1/p}, \quad 1 \le p < \infty,$$

$$||\phi||_{L^\infty(0,T;X)} = \text{ess} \sup_{0<t<T} ||\phi(t)||_X.$$

Finally, we introduce the space $\ell^\infty(0,T;X)$ defined by

$$\ell^\infty(0,T;X) := \left\{ f : \{t^0, \dots, t^M\} \to X | \ \ ||f||_{\ell^\infty(0,T;X)} = \max_{1\le n\le M} ||f(t^n)||_X < \infty \right\},$$

where X denotes a functional space (in our context X should be L^p, $p \in [1,\infty]$), and the space $L^{1,\infty}$ defined by

$$L^{1,\infty} = \left\{ f \in L^1 \cap L^\infty \ | \ \|f\|_{L^{1,\infty}} < \infty \right\},$$

where

$$\|f\|_{L^{1,\infty}} = \|f\|_{L^1} + \|f\|_{L^\infty}.$$

2.2. Existence, uniqueness, and regularity of the solution of the continuous problem. In this section we recall a theorem of existence of a classical solution for the Vlasov–Poisson system. The following theorem gives the existence, uniqueness, and regularity of the classical solutions, global in time, of the Vlasov–Poisson periodic system in one dimension.

THEOREM 2.1. *Assuming $f_0 \in \mathscr{C}^1_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)$, positive, periodic with respect to the variable x with period L, and $Q(0) \leq R$ with $R > 0$ and $Q(t)$ defined as*

$$Q(t) = 1 + \sup\{|v| : \exists x \in [0, L], \quad \tau \in [0, t] \mid f(\tau, x, v) \neq 0\}$$

and

$$\frac{1}{L}\int_0^L \int_{-\infty}^{+\infty} f_0(x, v) dv dx = 1,$$

then the periodic Vlasov–Poisson system has a unique classical solution (f, E), periodic in x, with period L, for all time t in $[0, T]$, such that

$$f \in \mathscr{C}^1_b\left(0, T; \mathscr{C}^1_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)\right),$$
$$E \in \mathscr{C}^1_b\left(0, T; \mathscr{C}^1_{b,per_x}(\mathbb{R})\right),$$

and there exists a constant $C = C(R, f_0)$ dependent on R and f_0 such that

$$Q(T) \leq CT.$$

Moreover, if we assume $f_0 \in \mathscr{C}^m_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)$, then $(f, E) \in \mathscr{C}^m_b(0, T; \mathscr{C}^m_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)) \times \mathscr{C}^m_b(0, T; \mathscr{C}^m_{b,per_x}(\mathbb{R}))$ for all finite time T.

Proof. We do not write out the proof because it is a straightforward adaptation of the proof done by Schaeffer in [38]. We refer the reader to the articles [34, 28, 29, 23, 15, 30, 31]. ∎

2.3. Regularity assumptions for the continuous problem. For our purpose, we first suppose that $f_0(x, v)$ satisfies the following regularity assumptions:

$$f_0 \in \mathscr{C}^2_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v).$$

Then, as is proven in Glassey [23], if f_0 is smooth and compactly supported, the solution of the Vlasov–Poisson system remains smooth and compactly supported for all time. Theorem 2.1 gives the existence and uniqueness of the solution (f, E) such that

$$f \in \mathscr{C}^2_b\left(0, T; \mathscr{C}^2_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)\right), \tag{2.11}$$
$$E \in \mathscr{C}^2_b\left(0, T; \mathscr{C}^2_{b,per_x}(\mathbb{R})\right). \tag{2.12}$$

Further, we prove that we still have convergence under weaker regularity assumptions.

3. The discrete problem.

3.1. Space of approximation and the interpolation operator. Let $Q = [0, L] \times \mathbb{R}$, $\Omega = [0, L] \times [-R, R]$ with $R > 0$, and $\mathcal{T}_h$ be a triangulation of Q.

Before going further we impose some regularity assumptions on the triangulation $\mathcal{T}_h$ as follows:

(H1) The triangulation $\mathcal{T}_h$ is regular; that is to say, there exists a constant σ such that

$$\frac{h_T}{\rho_T} \leq \sigma \quad \forall T \in \mathcal{T}_h,$$

and the quantity $h = \max_{\{T\in\mathcal{T}_h\}} h_T$ approaches zero, where h_T and ρ_T denote, respectively, the exterior and the interior diameter of a finite element T.

(H2) All the finite elements (T, P_T, Σ_T), $T \in \mathcal{T}_h$, are affine equivalent to a single reference finite element $(\widehat{T}, \widehat{P}, \widehat{\Sigma})$ (see [14]).

Let P_m be the space Lagrange polynomial of degree less than or equal to m, and let X_h be the space defined by

$$X_h = \{g \in W^{1,\infty} \cap W^{1,p}(Q),\ g_{|T} \in P_m\ \forall T \in \mathcal{T}_h\}.$$

Let π_h be a continuous linear interpolation operator from $W^{m+1,\infty} \cap W^{m+1,p}(Q)$, $1 \leq p < \infty$, onto X_h. The interpolation error estimations in Sobolev spaces (see [14]) give, with $k \in \{0, 1\}$ and $q \in \{p, \infty\}$,

$$||f - \pi_h f||_{W^{k,q}(Q)} \leq Ch^{m+1-k}|f|_{W^{m+1,q}} \quad \forall f \in W^{m+1,\infty} \cap W^{m+1,q}(Q). \tag{3.1}$$

The space X_h is characterized by its basis functions, denoted by $\{\psi_k\}$.

3.2. Transport operators. Now we introduce some transport operators. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the operators defined as

$$\mathcal{T}_1 g(t, x, v) = g\left(t, x - v\frac{\Delta t}{2}, v\right),$$

$$\mathcal{T}_2 g(t, x, v) = g(t, x, v - \Delta t\widetilde{E}(t, x)),$$

where $\widetilde{E}(t, x)$ is the solution of the following problem:

$$\begin{cases} \dfrac{d\widetilde{E}}{dx}(t, x) = \displaystyle\int_v \mathcal{T}_1 g(t, x, v)dv - 1, \\ \displaystyle\int_0^L \widetilde{E}(t, x)dx = 0. \end{cases} \tag{3.2}$$

Let $\widetilde{\mathcal{T}}_1$ be the transport operator defined as

$$\widetilde{\mathcal{T}}_1 g(t, x, v) = \pi_h g\left(t, x - v\frac{\Delta t}{2}, v\right),$$

where

$$\pi_h g(t, x, v) = \sum_k g(t, x_k, v_k)\psi_k(x, v),$$

and let $\widetilde{\mathcal{T}}_2$ be defined as

$$\widetilde{\mathcal{T}}_2 g(t, x, v) = \pi_h g(t, x, v - \Delta t\widetilde{E}(t, x)).$$

Finally we introduce

$$\widetilde{\mathcal{T}}_2^{\star} g(t,x,v) = \pi_h g(t,x,v-\Delta t E_h(t,x)),$$

where $E_h(t,x)$ is the solution of the following problem:

$$\text{(3.3)} \qquad \begin{cases} \dfrac{dE_h}{dx}(t,x) = \displaystyle\int_v \widetilde{\mathcal{T}}_1 g(t,x,v)dv - 1, \\ \displaystyle\int_0^L E_h(t,x)dx = 0. \end{cases}$$

Notice that (2.7) implies that $\widetilde{E}(t,x)$ and $E_h(t,x)$ are respectively given by

$$\widetilde{E}(t,x) = \int_0^L K(x,y)\left(\int_{-\infty}^{+\infty} \mathcal{T}_1 g(t,y,v)dv - 1\right)dy$$

and

$$E_h(t,x) = \int_0^L K(x,y)\left(\int_{-\infty}^{+\infty} \widetilde{\mathcal{T}}_1 g(t,y,v)dv - 1\right)dy.$$

4. The numerical scheme. We suppose that we know $f_h(t^n)$ defined on $\mathcal{T}_h$. Therefore the numerical scheme which allows us to go from time t^n to t^{n+1} and compute $f_h(t^{n+1})$ can be described in four steps:

(A1) We evaluate the distribution at time t^n at the foot of the field-free characteristics starting at (x,v) at time $t^{n+1/2}$ using a Lagrange interpolation operator. This action is described by the transport operator $\widetilde{\mathcal{T}}_1$.

(A2) The output from (A1) is integrated with respect to velocity to provide an approximation for the density at time $t^{n+1/2}$, which is then substituted into the Poisson equation (3.3) to compute the approximation of the electric field at time $t^{n+1/2}$.

(A3) The result obtained from (A1) is evaluated at the foot of the velocity characteristic starting at (x,v) at time t^{n+1} with the acceleration field found in (A2) using a Lagrange interpolation operator. This action is described by the transport operator $\widetilde{\mathcal{T}}_2^{\star}$

(A4) Between time $t^{n+1/2}$ and t^{n+1}, we apply step (A1) to the output from (A3). This action is described by the transport operator $\widetilde{\mathcal{T}}_1$. Then we obtain $f_h(t^{n+1})$, which is the new initial data for the algorithm (A1)–(A4).

Using transport operators defined above in section 3.2, the numerical scheme can be written as

$$f_h(t^{n+1},x,v) = \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2^{\star} \circ \widetilde{\mathcal{T}}_1 f_h(t^n,x,v),$$

where $f_h(0,x,v) = \pi_h f_0(x,v)$ is a discretization of f_0 for the initial data,

$$f_h(t^n,x+L,v) = f_h(t^n,x,v) \quad \forall |v| \le Q(T)$$

is the boundary condition in x, and

$$f_h(t^n,x,v) = 0 \quad \forall |v| > Q(T),\ \forall x \in [0,L]$$

is the boundary condition in v.

5. Convergence analysis.

5.1. Main theorem. We next give the convergence theorem.

THEOREM 5.1. *Assuming $f_0 \in \mathscr{C}^2_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)$, positive, periodic with respect to the variable x with period L, then the numerical solution of the Vlasov–Poisson system (f_h, E_h), computed by the numerical scheme exposed in section 4, converges toward the solution (f, E) of the periodic Vlasov–Poisson system, and there exists a constant $C = C(||f||_{\mathscr{C}^2(0,T;W^{2,\infty}(Q))})$ independent of Δt, h such that*

$$||f - f_h||_{\ell^\infty(0,T;L^\infty(Q))} \le C\left(||f||_{\mathscr{C}^2(0,T;W^{2,\infty}(Q))}\right)\left(\Delta t^2 + h^2 + \frac{h^2}{\Delta t}\right)$$

and

$$||E - E_h||_{\ell^\infty(0,T;L^\infty([0,L]))} \le C\left(||f||_{\mathscr{C}^2(0,T;W^{2,\infty}(Q))}\right)\left(\Delta t^2 + h^2 + \frac{h^2}{\Delta t}\right).$$

Remark 5.2. In Theorem 5.1 we have a lot of choices for the time step. We note that the convergence rate is slightly better than first order: If we make the choice $\Delta t = h^{2/3}$, then the error estimate involves $h^{4/3}$ rather than h to the first power. Therefore we see that the main reason for using semi-Lagrangian schemes in lieu of particle schemes comes from the nice flexibility of the error estimates stated in Theorem 5.1, because they allow us to choose larger time steps and get convergence rates higher than one.

5.2. Idea of the proof. We want to evaluate the global error at time t^{n+1}:

$$e^{n+1} = ||f(t^{n+1}, x, v) - f_h(t^{n+1}, x, v)||_{L^\infty(Q)}.$$

Therefore we decompose $f(t^{n+1}, x, v) - f_h(t^{n+1}, x, v)$ as

$$\begin{aligned} f(t^{n+1}, x, v) - f_h(t^{n+1}, x, v) &= f(t^{n+1}, x, v) - \mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n, x, v) \\ &\quad + \mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n, x, v) - \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2 \circ \widetilde{\mathcal{T}}_1 f(t^n, x, v) \\ &\quad + \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2 \circ \widetilde{\mathcal{T}}_1 f(t^n, x, v) - \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2^\star \circ \widetilde{\mathcal{T}}_1 f(t^n, x, v) \\ &\quad + \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2^\star \circ \widetilde{\mathcal{T}}_1 (f(t^n, x, v) - f_h(t^n, x, v)). \end{aligned}$$

In order to estimate e^{n+1} we will estimate the four terms on the right-hand side of this equation. These estimations are described in the following section.

5.3. A priori estimates. We begin with the following lemma, which gives an estimate of the time discretization error.

LEMMA 5.3. *Assume that $f \in \mathscr{C}^2_b(0, T; \mathscr{C}^2_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v))$; then there exists a constant C such that*

$$\left\| f(t^{n+1}) - \mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n) \right\|_{L^\infty(Q)} \le C\left(\|f\|_{\mathscr{C}^2(0,T;W^{2,\infty}(Q))}\right)\Delta t^3.$$

Proof. As f is constant along the characteristic curves, we have

$$\begin{aligned} f(t^{n+1}, x, v) &= f(t^{n+1}, X(t^{n+1}; t^{n+1}, x, v), V(t^{n+1}; t^{n+1}, x, v)) \\ &= f(t^n, X(t^n; t^{n+1}, x, v), V(t^n; t^{n+1}, x, v)) \\ &= f(t^n, X(t^n), V(t^n)), \end{aligned}$$

where $X(t^n) = X(t^n; t^{n+1}, x, v)$ and $V(t^n) = V(t^n; t^{n+1}, x, v)$. On the other hand, we have

$$\begin{aligned}
\mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n) &= \mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n, x, v) \\
&= \mathcal{T}_1 \circ \mathcal{T}_2 f\Big(t^n, x - v\tfrac{\Delta t}{2}, v\Big) \\
&= \mathcal{T}_1 f\left(t^n, x - v\tfrac{\Delta t}{2} + \tfrac{\Delta t^2}{2}\widetilde{E}(t^{n+1/2}, x), v - \Delta t \widetilde{E}(t^{n+1/2}, x)\right) \\
&= f\Big(t^n, x - v\Delta t + \tfrac{\Delta t^2}{2}\widetilde{E}\Big(t^{n+1/2}, x - v\tfrac{\Delta t}{2}\Big), v - \Delta t\widetilde{E}\Big(t^{n+1/2}, x - v\tfrac{\Delta t}{2}\Big)\Big) \\
&= f(t^n, \widetilde{X}(t^n; t^{n+1}, x, v), \widetilde{V}(t^n; t^{n+1}, x, v)) \\
&= f(t^n, \widetilde{X}(t^n), \widetilde{V}(t^n)),
\end{aligned}$$

where

$$\widetilde{X}(t^n) = x - v\Delta t + \frac{\Delta t^2}{2}\widetilde{E}\left(t^{n+1/2}, x - v\frac{\Delta t}{2}\right)$$

and

$$\widetilde{V}(t^n) = v - \Delta t \widetilde{E}\left(t^{n+1/2}, x - v\frac{\Delta t}{2}\right).$$

In order to justify the following Taylor expansion, we remember that assumption (2.12) gives $E \in \mathscr{C}_b^2(0, T; \mathscr{C}^2_{b,per_x}(\mathbb{R}))$. We notice that $\widetilde{E}$ has the same regularity in space as E, as the source terms in Poisson equations (3.2) and (2.2) also have the same regularity.

Hence a Taylor expansion gives

$$\begin{aligned}
X(t^{n+1/2}) - \big(x - v\tfrac{\Delta t}{2}\big) &= X(t^{n+1/2}) - \big(X(t^{n+1}) - V(t^{n+1})\tfrac{\Delta t}{2}\big) \\
&= X(t^{n+1/2}) - \big(X(t^{n+1}) - \tfrac{\Delta t}{2}\dot{X}(t^{n+1})\big) \\
&= O(\Delta t^2).
\end{aligned} \tag{5.1}$$

As $f \in \mathscr{C}_b^2(0, T; \mathscr{C}^2_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v))$, we have

$$\begin{aligned}
\frac{f(t^{n+1/2}, x, v) - f\big(t^n, x - v\frac{\Delta t}{2}, v\big)}{\frac{\Delta t}{2}} &= \partial_t f(t^{n+1/2}, x, v) + v\partial_x f(t^{n+1/2}, x, v) + O(\Delta t) \\
&= -E(t^{n+1/2}, x)\partial_v f(t^{n+1/2}, x, v) + O(\Delta t).
\end{aligned} \tag{5.2}$$

Then, using (2.7) and (5.2), we get

$$\begin{aligned}
&E(t^{n+1/2}, x) - \widetilde{E}(t^{n+1/2}, x) \\
&\quad = \int_0^L K(x, y)\left(\int_{-\infty}^{+\infty}\Big[f(t^{n+1/2}, y, v) - f\Big(t^n, y - v\tfrac{\Delta t}{2}, v\Big)\Big]dv\right)dy \\
&\quad \leq C\left(\|f\|_{\mathscr{C}^2(0,T;W^{2,\infty}(Q))}\right)\Delta t^2.
\end{aligned} \tag{5.3}$$

Using (5.1) and (5.3), we obtain

$$
\begin{aligned}
V(t^n)-\widetilde{V}(t^n) &= V(t^n)-\Big(V(t^{n+1})-\Delta t\widetilde{E}\Big(t^{n+1/2},X(t^{n+1})-V(t^{n+1})\tfrac{\Delta t}{2}\Big)\Big)\\
&= V(t^n)-\Big(V(t^{n+1})-\Delta t\widetilde{E}\left(t^{n+1/2},X(t^{n+1/2})+O(\Delta t^2)\right)\Big)\\
&= V(t^n)-\left(V(t^{n+1})-\Delta t E\left(t^{n+1/2},X(t^{n+1/2})+O(\Delta t^2)\right)\right)\\
&\quad +\Delta t\Big(\widetilde{E}\ \left(t^{n+1/2},X(t^{n+1/2})+O(\Delta t^2)\right)\\
&\qquad\quad -E\left(t^{n+1/2},X(t^{n+1/2})+O(\Delta t^2)\right)\Big)\\
&= V(t^n)-\left(V(t^{n+1})-\Delta t E(t^{n+1/2},X(t^{n+1/2}))\right)+O(\Delta t^3)\\
&= V(t^n)-V(t^{n+1})+\Delta t\dot{V}(t^{n+1/2})+O(\Delta t^3)\\
&\leq C\left(\|f\|_{\mathscr{C}^2(0,T;W^{2,\infty}(Q))}\right)\Delta t^3
\end{aligned}
$$

and

$$
\begin{aligned}
X(t^n)-\widetilde{X}(t^n) &= X(t^n)-\Big(X(t^{n+1})-\Delta t V(t^{n+1})\\
&\qquad\qquad +\tfrac{\Delta t^2}{2}\widetilde{E}\Big(t^{n+1/2},X(t^{n+1})-V(t^{n+1})\tfrac{\Delta t}{2}\Big)\Big)\\
&= X(t^n)-\Big(X(t^{n+1})-\Delta t V(t^{n+1})\\
&\qquad\qquad +\tfrac{\Delta t^2}{2}\widetilde{E}\big(t^{n+1/2},X(t^{n+1/2})+O(\Delta t^2)\big)\Big)\\
&= X(t^n)-\Big(X(t^{n+1})-\Delta t V(t^{n+1})\\
&\qquad\qquad +\tfrac{\Delta t^2}{2}E\big(t^{n+1/2},X(t^{n+1/2})+O(\Delta t^2)\big)\Big)\\
&\quad -\tfrac{\Delta t^2}{2}\Big(\widetilde{E}(t^{n+1/2},X(t^{n+1/2})+O(\Delta t^2))\\
&\qquad\quad -E(t^{n+1/2},X(t^{n+1/2})+O(\Delta t^2))\Big)\\
&= X(t^n)-\Big(X(t^{n+1})-\Delta t V(t^{n+1})\\
&\qquad\qquad +\tfrac{\Delta t^2}{2}E(t^{n+1/2},X(t^{n+1/2}))\Big)+O(\Delta t^4)\\
&= X(t^n)-\Big(X(t^{n+1})-\Delta t\dot{X}(t^{n+1})+\tfrac{\Delta t^2}{2}\ddot{X}(t^{n+1/2})\Big)+O(\Delta t^4)\\
&= X(t^n)-\Big(X(t^{n+1})-\Delta t\dot{X}(t^{n+1})+\tfrac{\Delta t^2}{2}\ddot{X}(t^{n+1})\Big)+O(\Delta t^3)\\
&\leq C\left(\|f\|_{\mathscr{C}^2(0,T;W^{2,\infty}(Q))}\right)\Delta t^3.
\end{aligned}
$$

Finally, we deduce that

$$
\begin{aligned}
\mathcal{T}_1\circ\mathcal{T}_2\circ\mathcal{T}_1 f(t^n) &= f(t^n,X(t^n)+O(\Delta t^3),V(t^n)+O(\Delta t^3))\\
&= f(t^n,X(t^n),V(t^n))+\nabla f(t^n,X(t^n),V(t^n))\cdot O(\Delta t^3)\\
&= f(t^{n+1},X(t^{n+1}),V(t^{n+1}))+\nabla f(t^n,X(t^n),V(t^n))\cdot O(\Delta t^3)\\
&= f(t^{n+1},x,v)+\nabla f(t^n,X(t^n),V(t^n))\cdot O(\Delta t^3)
\end{aligned}
$$

and

$$||f(t^{n+1}) - \mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n)||_{L^p(Q)} \leq C\left(\|f\|_{\mathscr{C}^2(0,T;W^{2,\infty}(Q))}\right)||\nabla f||_{L^\infty([0,T]\times Q)}\Delta t^3. \qquad \square$$

We continue with the following result.

PROPOSITION 5.4. *Assume that* $f \in L^\infty(0,T;\mathscr{C}^{m+1}_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v))$, $m \geq 0$, *and* π_h *is a continuous linear interpolation operator from* $W^{m+1,\infty}(Q)$ *onto* X_h; *then there exists a constant* C *such that for* $i = 1, 2$, $1 \leq p \leq \infty$,

$$||\mathcal{T}_i f||_{L^\infty(0,T;W^{m+1,p}(Q))} \leq C||f||_{L^\infty(0,T;W^{m+1,p}(Q))}, \tag{5.4}$$

$$||\widetilde{\mathcal{T}}_i f||_{L^\infty(0,T;L^p(Q))} \leq C||f||_{L^\infty(0,T;W^{m+1,p}(Q))}, \tag{5.5}$$

and

$$\|(\mathcal{T}_i - \widetilde{\mathcal{T}}_i)f\|_{L^\infty(0,T;L^p(Q))} \leq Ch^{m+1}||f||_{L^\infty(0,T;W^{m+1,p}(Q))}. \tag{5.6}$$

Proof. It is obvious that

$$\left\|f\left(t, x - v\tfrac{\Delta t}{2}, v\right)\right\|_{L^\infty(0,T;L^p(Q))} = \|f\|_{L^\infty(0,T;L^p(Q))} \tag{5.7}$$

and

$$\|f(t, x, v - \widetilde{E}(t,x)\Delta t)\|_{L^\infty(0,T;L^p(Q))} = \|f\|_{L^\infty(0,T;L^p(Q))}\,. \tag{5.8}$$

On one side the gradient of $f(t, x - v\Delta t/2, v)$ gives

$$\left\|\partial_x\left(f\left(t, x - v\tfrac{\Delta t}{2}, v\right)\right)\right\|_{L^\infty(0,T;L^p(Q))} = \|\partial_x f\|_{L^\infty(0,T;L^p(Q))}$$

and

$$\left\|\partial_v\left(f(t, x - v\tfrac{\Delta t}{2}, v)\right)\right\|_{L^\infty(0,T;L^p(Q))} \leq \tfrac{\Delta t}{2}\,\|\partial_x f\|_{L^\infty(0,T;L^p(Q))} + \|\partial_v f\|_{L^\infty(0,T;L^p(Q))}\,.$$

Hence

$$\left\|f\left(t, x - v\tfrac{\Delta t}{2}, v\right)\right\|_{L^\infty(0,T;W^{1,p}(Q))} \leq C\,\|f\|_{L^\infty(0,T;W^{1,p}(Q))}\,.$$

In the same way we get

$$\left\|f\left(t, x - v\tfrac{\Delta t}{2}, v\right)\right\|_{L^\infty(0,T;W^{m+1,p}(Q))} \leq C\,\|f\|_{L^\infty(0,T;W^{m+1,p}(Q))}\,.$$

On the other side the gradient of $f(t, x, v - \widetilde{E}(t,x)\Delta t)$ gives

$$\begin{aligned}&\|\partial_x(f(t, x, v - \widetilde{E}(t,x)\Delta t))\|_{L^\infty(0,T;L^p(Q))}\\ &\qquad \leq \|\partial_x f\|_{L^\infty(0,T;L^p(Q))} + \Delta t||\partial_x \widetilde{E}||_{L^\infty([0,T]\times[0,L])}\,\|\partial_v f\|_{L^\infty(0,T;L^p(Q))}\end{aligned}$$

and

$$\|\partial_v(f(t, x, v - \widetilde{E}(t,x)\Delta t))\|_{L^\infty(0,T;L^p(Q))} \leq \|\partial_v f\|_{L^\infty(0,T;L^p(Q))}\,.$$

Hence

$$\begin{aligned}\|f(t,x,v-\widetilde{E}(t,x)\Delta t)\|_{L^\infty(0,T;W^{1,p}(Q))} &\leq (1+C\Delta t)\,\|f\|_{L^\infty(0,T;W^{1,p}(Q))}\\ &\leq C\,\|f\|_{L^\infty(0,T;W^{1,p}(Q))}\,.\end{aligned}$$

In the same way, as $\widetilde{E}\in L^\infty(0,T;\mathscr{C}^{m+1}_{b,per_x}(\mathbb{R}))$, we get

$$\|f(t,x,v-\widetilde{E}(t,x)\Delta t)\|_{L^\infty(0,T;W^{m+1,p}(Q))} \leq C\,\|f\|_{L^\infty(0,T;W^{m+1,p}(Q))}\,,$$

which completes the proof of (5.4).

π_h is an interpolation operator which is characterized by the basis functions $\{\psi_k\}$. Then $\pi_h f$ can be written as follows:

$$\pi_h f(t,x,v)=\sum_k f(t,x_k,v_k)\psi_k(x,v)=\sum_k f_k(t)\psi_k(x,v).$$

As any $\psi_k\in L^\infty(Q)$ and has compact support, there exists a constant M such that

$$\begin{aligned}\left\|\sum_k|\psi_k(x,v)|\right\|_{L^\infty(Q)} &\leq \sup_{T\in\mathcal{T}_h}\left\|\sum_k|\psi_k(x,v)|\right\|_{L^\infty(T)}\\ &\leq \operatorname{card}(\Sigma_T)\sup_{T\in\mathcal{T}_h}\max_{(x,v)\in T}|\psi_k(x,v)|\\ &\leq M,\end{aligned}$$

where Σ_T is the set of degrees of freedom on the triangle T.

- L^∞ case:

$$\|\pi_h f\|_{L^\infty(Q)}\leq\|f\|_{L^\infty(Q)}\sum_k|\psi_k(x,v)|\leq M\,\|f\|_{L^\infty(Q)}\,.$$

- L^1 case:

$$\int_Q|\pi_h f(t)|dvdx\leq\sum_k|f_k(t)|\int_Q|\psi_k|dxdv\leq M\sum_k|f_k(t)|\mathrm{meas}\left(\mathcal{S}_k\right),$$

where $\mathcal{S}_k$ is the support of ψ_k. Let $\mathcal{A}_k$ be the geometrical area associated with the node $N_k=(x_k,v_k)$, obtained by joining the barycenter of the triangles that have the vertex N_k in common to the middle of the edges of the triangles; then there exists a constant $K>0$ independent of h such that $(1/K)\mathrm{meas}(\mathcal{S}_k)\leq\mathrm{meas}(\mathcal{A}_k)<\mathrm{meas}(\mathcal{S}_k)$. Then we obtain

$$\|\pi_h f(t)\|_{L^1(Q)}\leq CMK\sum_k|f_k(t)|\mathrm{meas}\left(\mathcal{A}_k\right)\leq C\,\|f(t)\|_{L^1(Q)}$$

and

$$\|\pi_h f\|_{L^\infty([0,T],L^1(Q))}\leq C\,\|f\|_{L^\infty([0,T],L^1(Q))}\,.$$

- L^p case:

$$\int_Q|\pi_h f(t)|^p dvdx\leq\int_Q\left(\sum_k|f_k(t)|\,|\psi_k|\right)^p dvdx.$$

Thanks to the Hölder inequality, we get

$$\int_Q |\pi_h f(t)|^p \leq \int_Q \left(\sum_k |f_k(t)|^p |\psi_k|\right)\left(\sum_k |\psi_k|\right)^{p/p^*} dvdx,$$

with $p^* = p/(p-1)$. Then we get

$$\begin{aligned}\|\pi_h f(t)\|^p_{L^p(Q)} &\leq M^{p/p^*} \sum_k |f_k(t)|^p \int_Q |\psi_k| dxdv \\ &\leq KM^{p/p^*+1} \sum_k |f_k(t)|^p \text{meas}\,(\mathcal{A}_k) \\ &\leq C\,\|f\|^p_{L^p(Q)}\end{aligned}$$

and finally

$$\|\pi_h f\|_{L^\infty(0,T;L^p(Q))} \leq C\,\|f\|_{L^\infty(0,T;L^p(Q))}.$$

Hence, as $f \in L^\infty(0,T;\mathscr{C}^{m+1}_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v))$, then

$$\begin{aligned}\Big\|\pi_h f\Big(t, x - v\tfrac{\Delta t}{2}, v\Big)\Big\|_{L^\infty(0,T;L^p(Q))} &\leq C\Big\|f\Big(t, x - v\tfrac{\Delta t}{2}, v\Big)\Big\|_{L^\infty(0,T;L^p(Q))} \\ &\leq C||f||_{L^\infty(0,T;L^p(Q))} \\ &\leq C||f||_{L^\infty(0,T;W^{m+1,p}(Q))}\end{aligned}$$

and

$$\begin{aligned}\|\pi_h f(t,x,v - \widetilde{E}(t,x)\Delta t)\|_{L^\infty(0,T;L^p(Q))} &\leq C\|f(t,x,v - \widetilde{E}(t,x)\Delta t)\|_{L^\infty(0,T;L^p(Q))} \\ &\leq C||f||_{L^\infty(0,T;L^p(Q))} \\ &\leq C||f||_{L^\infty(0,T;W^{m+1,p}(Q))},\end{aligned}$$

which completes the proof of (5.5). Finally, thanks to inequality (3.1), we obtain

$$\begin{aligned}&\Big\|f\Big(t, x - v\tfrac{\Delta t}{2}, v\Big) - \pi_h f\Big(t, x - v\tfrac{\Delta t}{2}, v\Big)\Big\|_{L^\infty(0,T;L^p(Q))} \\ &\qquad \leq Ch^{m+1}\Big\|f\Big(t, x - v\tfrac{\Delta t}{2}, v\Big)\Big\|_{L^\infty(0,T;W^{m+1,p}(Q))} \\ &\qquad \leq Ch^{m+1}||f||_{L^\infty(0,T;W^{m+1,p}(Q))}\end{aligned}$$

and

$$\begin{aligned}&\|f(t,x,v - \widetilde{E}(t,x)\Delta t) - \pi_h f(t,x,v - \widetilde{E}(t,x)\Delta t)\|_{L^\infty(0,T;L^p(Q))} \\ &\qquad \leq Ch^{m+1}||f(t,x,v - \widetilde{E}(t,x)\Delta t)||_{L^\infty(0,T;W^{m+1,p}(Q))} \\ &\qquad \leq Ch^{m+1}||f||_{L^\infty(0,T;W^{m+1,p}(Q))},\end{aligned}$$

which completes the proof of the proposition. ∎

The next lemma gives an estimate of the space discretization error.

LEMMA 5.5. *Assume that $f \in L^\infty(0,T;\mathscr{C}^{m+1}_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v))$ and that π_h is a continuous linear interpolation operator from $W^{m+1,\infty}(Q)$ onto X_h; then there exists a constant C such that*

$$\|\mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n) - \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2 \circ \widetilde{\mathcal{T}}_1 f(t^n)\|_{L^\infty(Q)} \leq Ch^{m+1}||f||_{L^\infty(0,T;W^{m+1,\infty}(Q))}.$$

Proof. We begin with the following decomposition:

$$\begin{aligned} \mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n) - \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2 \circ \widetilde{\mathcal{T}}_1 f(t^n) = {}& (\mathcal{T}_1 - \widetilde{\mathcal{T}}_1) \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n) \\ &+ \widetilde{\mathcal{T}}_1 \circ (\mathcal{T}_2 - \widetilde{\mathcal{T}}_2) \circ \mathcal{T}_1 f(t^n) \\ &+ \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2 \circ (\mathcal{T}_1 - \widetilde{\mathcal{T}}_1) f(t^n). \end{aligned} \tag{5.9}$$

Using (5.4), (5.5), and (5.6), the decomposition (5.9) gives for the first term

$$\begin{aligned} ||(\mathcal{T}_1 - \widetilde{\mathcal{T}}_1) \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n)||_{L^\infty(\Omega)} &\leq Ch^{m+1}|\mathcal{T}_2 \circ \mathcal{T}_1 f(t^n)|_{W^{m+1,\infty}(Q)} \\ &\leq Ch^{m+1}|\mathcal{T}_1 f(t^n)|_{W^{m+1,\infty}(Q)} \\ &\leq Ch^{m+1}|f(t^n)|_{W^{m+1,\infty}(Q)} \\ &\leq Ch^{m+1}||f||_{L^\infty(0,T;W^{m+1,\infty}(Q))}, \end{aligned}$$

for the second term of (5.9)

$$\begin{aligned} ||\widetilde{\mathcal{T}}_1 \circ (\mathcal{T}_2 - \widetilde{\mathcal{T}}_2) \circ \mathcal{T}_1 f(t^n)||_{L^\infty(Q)} &\leq C||(\mathcal{T}_2 - \widetilde{\mathcal{T}}_2) \circ \mathcal{T}_1 f(t^n)||_{L^\infty(Q)} \\ &\leq Ch^{m+1}|\mathcal{T}_1 f(t^n)|_{W^{m+1,\infty}(Q)} \\ &\leq Ch^{m+1}||f||_{L^\infty(0,T;W^{m+1,\infty}(Q))}, \end{aligned}$$

and for the third term of (5.9)

$$\begin{aligned} ||\widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2 \circ (\mathcal{T}_1 - \widetilde{\mathcal{T}}_1) f(t^n)||_{L^\infty(Q)} &\leq C||(\mathcal{T}_1 - \widetilde{\mathcal{T}}_1) f(t^n)||_{L^\infty(Q)} \\ &\leq Ch^{m+1}||f||_{L^\infty(0,T;W^{m+1,\infty}(Q))}, \end{aligned}$$

which proves the lemma. ◻

We continue with the proof of another lemma that gives an estimate of a coupling error between the resolution of the Vlasov and the Poisson equations.

LEMMA 5.6. *Assume that $f \in L^\infty(0,T;\mathscr{C}^{m+1}_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v))$ and that π_h is a continuous linear interpolation operator from $W^{m+1,\infty}(Q)$ onto X_h; then there exists a constant C such that*

$$\|\widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2 \circ \widetilde{\mathcal{T}}_1 f(t^n) - \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2^\star \circ \widetilde{\mathcal{T}}_1 f(t^n)\|_{L^\infty(Q)} \leq C\Delta t \left(e^n + h^{m+1}\right) ||f||_{L^\infty(0,T;W^{m+1,\infty}(Q))},$$

where

$$e^n = \|f(t^n) - f_h(t^n)\|_{L^\infty(Q)}.$$

Proof. On the one hand, we have

$$(\widetilde{\mathcal{T}}_2 - \widetilde{\mathcal{T}}_2^\star)g(t^n) = \pi_h\left(g(t^n, x, v - \Delta t\widetilde{E}^{n+1/2}(x)) - g(t^n, x, v - \Delta t E_h^{n+1/2}(x))\right).$$

On the other hand, we have

$$\begin{aligned} &|g(t^n, x, v - \Delta t\widetilde{E}^{n+1/2}(x)) - g(t^n, x, v - \Delta t E_h^{n+1/2}(x))| \\ &\qquad \leq \Delta t|\widetilde{E}^{n+1/2}(x) - E_h^{n+1/2}(x)| \, \|\nabla g(t^n)\|_{L^\infty(Q)}, \end{aligned}$$

where $\widetilde{E}^{n+1/2}(x)$ and $E_h^{n+1/2}(x)$ can be written as follows:

$$\widetilde{E}^{n+1/2}(x) = \int_0^L K(x,y)\left(\int_{\mathbb{R}} \mathcal{T}_1 f(t^n,y,v)dv - 1\right)dy,$$
$$E_h^{n+1/2}(x) = \int_0^L K(x,y)\left(\int_{\mathbb{R}} \widetilde{\mathcal{T}}_1 f_h(t^n,y,v)dv - 1\right)dy.$$

Then we can write

$$\begin{aligned} E_h^{n+1/2}(x) - \widetilde{E}^{n+1/2}(x) &= \int_0^L K(x,y)\left(\int_{\mathbb{R}}\left[\widetilde{\mathcal{T}}_1 f_h(t^n,y,v) - \mathcal{T}_1 f(t^n,y,v)\right]dv\right)dy, \\ &= \int_0^L K(x,y)\left(\int_{|v|\le Q(T)} \pi_h\left[f_h\left(t^n, y - v\tfrac{\Delta t}{2}, v\right) - f\left(t^n, y - v\tfrac{\Delta t}{2}, v\right)\right]dv\right)dy \\ &\quad + \int_0^L\int_{|v|\le Q(T)} K(x,y)\left(\pi_h f\left(t^n, y - v\tfrac{\Delta t}{2}, v\right) - f\left(t^n, y - v\tfrac{\Delta t}{2}, v\right)\right)dvdy, \end{aligned}$$

so that we get

$$\begin{aligned} &\|E_h^{n+1/2} - \widetilde{E}^{n+1/2}\|_{L^\infty([0,L])} \\ &\qquad \le LQ(T)||K||_{L^\infty}\left\|\pi_h\left[f_h\left(t^n, y - v\tfrac{\Delta t}{2}, v\right) - f\left(t^n, y - v\tfrac{\Delta t}{2}, v\right)\right]\right\|_{L^\infty(Q)} \\ &\qquad\quad + LQ(T)||K||_{L^\infty}\left\|\pi_h f\left(t^n, y - v\tfrac{\Delta t}{2}, v\right) - f\left(t^n, y - v\tfrac{\Delta t}{2}, v\right)\right\|_{L^\infty(Q)}, \end{aligned} \tag{5.10}$$

and using (5.4), (5.5), and (5.6),

$$\begin{aligned} \|E_h^{n+1/2} - \widetilde{E}^{n+1/2}\|_{L^\infty([0,L])} &\le LQ(T)||K||_{L^\infty}||\pi_h||_{L^\infty}\,\|f_h(t^n) - f(t^n)\|_{L^\infty(Q)} \\ &\quad + CLQ(T)||K||_{L^\infty}h^{m+1}||f(t^n)||_{W^{m+1,\infty}(Q)}. \end{aligned} \tag{5.11}$$

Finally, we obtain

$$\|E_h^{n+1/2} - \widetilde{E}^{n+1/2}\|_{L^\infty([0,L])} \le C\left(e^n + h^{m+1}\right)$$

and, as a consequence,

$$\|(\widetilde{\mathcal{T}}_2 - \widetilde{\mathcal{T}}_2^\star)g(t^n)\|_{L^\infty(Q)} \le C\Delta t\left(e^n + h^{m+1}\right)\|\nabla g(t^n)\|_{L^\infty(Q)}. \tag{5.12}$$

Then, using (5.4) and (5.12),

$$\begin{aligned} \|\widetilde{\mathcal{T}}_1 \circ (\widetilde{\mathcal{T}}_2 - \widetilde{\mathcal{T}}_2^\star) \circ \widetilde{\mathcal{T}}_1 f(t^n)\|_{L^\infty(Q)} &\le C\|(\widetilde{\mathcal{T}}_2 - \widetilde{\mathcal{T}}_2) \circ \widetilde{\mathcal{T}}_1 f(t^n)\|_{L^\infty(Q)} \\ &\le C\Delta t\left(e^n + h^{m+1}\right)\|\nabla(\widetilde{\mathcal{T}}_1 f(t^n))\|_{L^\infty(Q)}. \end{aligned}$$

Now we estimate the term $\|\nabla(\widetilde{\mathcal{T}}_1 f(t^n))\|_{L^\infty(Q)}$. We can do this in the following way. Using (3.1), we get

$$\begin{aligned}
\|\nabla(\widetilde{\mathcal{T}}_1 f(t^n))\|_{L^\infty(Q)} &\le \left\|\nabla\Big(\pi_h f\Big(t^n, x - v\tfrac{\Delta t}{2}, v\Big)\Big)\right\|_{L^\infty(Q)} \\
&\le \left\|\nabla\Big[(\pi_h f - f)\Big(t^n, x - v\tfrac{\Delta t}{2}, v\Big)\Big]\right\|_{L^\infty(Q)} \\
&\qquad + \left\|\nabla\Big(f\Big(t^n, x - v\tfrac{\Delta t}{2}, v\Big)\Big)\right\|_{L^\infty(Q)} \\
&\le Ch^m ||f||_{L^\infty(0,T;W^{m+1,\infty}(Q))} + ||f||_{L^\infty(0,T;W^{m+1,\infty}(Q))} \\
&\le C||f||_{L^\infty(0,T;W^{m+1,\infty}(Q))}.
\end{aligned}$$

In fact, this estimation is due to the continuity of π_h from $W^{m+1,\infty}(Q)$ onto X_h. Then we finally obtain

$$\|\widetilde{\mathcal{T}}_1 \circ (\widetilde{\mathcal{T}}_2 - \widetilde{\mathcal{T}}_2^\star) \circ \widetilde{\mathcal{T}}_1 f(t^n)\|_{L^\infty(Q)} \le C\Delta t\left(e^n + h^{m+1}\right) ||f||_{L^\infty(0,T;W^{m+1,\infty}(Q))},$$

which completes the proof. $\square$

We now state the last lemma, which gives information about the stability of the numerical scheme.

LEMMA 5.7. *Let π_h be the interpolation operator from $W^{2,\infty}(Q)$ onto X_h with $P_m = P_1$; then we have*

$$\|\widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2^\star \circ \widetilde{\mathcal{T}}_1(f(t^n) - f_h(t^n))\|_{L^\infty(Q)} \le e^n. \tag{5.13}$$

Proof. As π_h is a linear interpolation operator, the basis functions satisfy

$$0 \le \psi_k \le 1$$

and

$$\sum_k \psi_k = 1,$$

and therefore we have

$$\|\pi_h\|_{L^\infty} = \sup_{\substack{f\in L^\infty \\ f\neq 0}} \frac{\|\pi_h f\|_{L^\infty(Q)}}{\|f\|_{L^\infty(Q)}} \le 1.$$

Indeed we have

$$\begin{aligned}
|\pi_h g| &= \left|\sum_k g(x_k, v_k)\psi_k(x,v)\right| \\
&\le \sum_k |g_k|\psi_k(x,v) \\
&\le \|g\|_{L^\infty} \sum_k \psi_k = \|g\|_{L^\infty}.
\end{aligned}$$

As a consequence we obviously obtain

$$\begin{aligned}
\|\widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2^\star \circ \widetilde{\mathcal{T}}_1(f(t^n) - f_h(t^n))\|_{L^\infty(Q)} &\le \|\widetilde{\mathcal{T}}_2^\star \circ \widetilde{\mathcal{T}}_1(f(t^n) - f_h(t^n))\|_{L^\infty(Q)} \\
&\le \|\widetilde{\mathcal{T}}_1(f(t^n) - f_h(t^n))\|_{L^\infty(Q)} \\
&\le \|f(t^n) - f_h(t^n)\|_{L^\infty(Q)},
\end{aligned}$$

which completes the proof. $\square$

Now we can return to the proof of the main theorem.

Proof of the main theorem. We want to evaluate the global error at time t^{n+1}:

$$e^{n+1} = ||f(t^{n+1}, x, v) - f_h(t^{n+1}, x, v)||_{L^\infty(Q)}.$$

We decompose $f(t^{n+1}, x, v) - f_h(t^{n+1}, x, v)$ as

$$\begin{aligned} f(t^{n+1}, x, v) - f_h(t^{n+1}, x, v) = {} & f(t^{n+1}, x, v) - \mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n, x, v) \\ & + \mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n, x, v) - \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2 \circ \widetilde{\mathcal{T}}_1 f(t^n, x, v) \\ & + \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2 \circ \widetilde{\mathcal{T}}_1 f(t^n, x, v) - \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2^\star \circ \widetilde{\mathcal{T}}_1 f(t^n, x, v) \\ & + \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2^\star \circ \widetilde{\mathcal{T}}_1 (f(t^n, x, v) - f_h(t^n, x, v)). \end{aligned} \tag{5.14}$$

Finally if we put together Lemmas 5.3, 5.5, 5.6, 5.7, we obtain the following estimation:

$$e^{n+1} \leq (1 + C\Delta t) e^n + C\left(||f||_{\mathscr{C}^2(0,T;W^{2,\infty}(Q))}\right)\left(\Delta t^3 + h^2 + h^2 \Delta t\right).$$

A discrete Gronwall inequality enables us to get

$$e^{n+1} \leq \exp(CT) e^0 + C\left(||f||_{\mathscr{C}^2(0,T;W^{2,\infty}(Q))}\right)\left(\Delta t^2 + \frac{h^2}{\Delta t} + h^2\right).$$

As e^0 is only a fixed interpolation error, we obtain

$$e^n \leq C\left(||f||_{\mathscr{C}^2(0,T;W^{2,\infty}(Q))}\right)\left(\Delta t^2 + \frac{h^2}{\Delta t} + h^2\right).$$

In order to prove the convergence of the electric field, we estimate

$$||E(t^{n+1/2}) - E_h^{n+1/2}||_{L^\infty([0,L])}.$$

To estimate this term we proceed as in the proof of Lemmas 5.6 and 5.3. Then we obtain

$$||\widetilde{E}(t^{n+1/2}) - E_h^{n+1/2}||_{L^\infty([0,L])} \leq C\left(||f||_{\mathscr{C}^2(0,T;W^{2,\infty}(Q))}\right)\left(\Delta t^2 + h^2 + \frac{h^2}{\Delta t}\right)$$

and

$$||E(t^{n+1/2}) - \widetilde{E}(t^{n+1/2})||_{L^\infty([0,L])} \leq C\left(||f||_{\mathscr{C}^2(0,T;W^{2,\infty}(Q))}\right)\Delta t^2$$

so that

$$||E(t^{n+1/2}) - E_h^{n+1/2}||_{L^\infty([0,L])} \leq C\left(||f||_{\mathscr{C}^2(0,T;W^{2,\infty}(Q))}\right)\left(\Delta t^2 + h^2 + \frac{h^2}{\Delta t}\right). \qquad \square$$

5.4. Other results. We can prove the convergence of our numerical scheme under weaker regularity assumptions. Following the proof of existence and uniqueness of the solutions of the Cauchy problem for the Vlasov–Maxwell system in one dimension made by Cooper and Klimas [15], if we take f_0 such that

$$f_0 \in \mathscr{C}_{b,per_x} \cap W_c^{1,\infty}(\mathbb{R}_x \times \mathbb{R}_v),$$

the Vlasov–Poisson periodic system given by (2.7), (2.8), (2.9), and (2.10) has a unique solution (f,E) such that

$$f \in \mathscr{C}_b\left(0,T;\mathscr{C}_{b,per_x} \cap W_c^{1,\infty}(\mathbb{R}_x\times\mathbb{R}_v)\right),$$
$$\partial_t f \in L^\infty\left(0,T;L_c^\infty(\mathbb{R}_x\times\mathbb{R}_v)\right),$$

where the derivative is taken in the sense of distribution, and

$$E \in \mathscr{C}^1\left(0,T;\mathscr{C}^1_{b,per_x}(\mathbb{R}_x)\right).$$

Now we state the theorem.

THEOREM 5.8. *Assume that $f_0 \in \mathscr{C}_{b,per_x} \cap W_c^{1,\infty}(\mathbb{R}_x\times\mathbb{R}_v)$. Let $\alpha > 0$, $h \sim \Delta t^{1/\varepsilon}$, with $0<\varepsilon<1$; then (f_h,E_h), the numerical solution of the periodic Vlasov–Poisson system, converges towards (f,E), and there exists a constant $C = C(\|f\|_{\mathscr{C}_b(0,T;W^{1,\infty}(Q))}, \|\partial_t f\|_{L^\infty(0,T;L^\infty(Q))})$ independent of Δt and h such that*

$$||f-f_h||_{\ell^\infty(0,T;L^\infty(Q))} \le C\left(\Delta t + h + h^{1-\varepsilon}\right)$$

and

$$||E-E_h||_{\ell^\infty(0,T;L^\infty([0,L]))} \le C\left(\Delta t + h + h^{1-\varepsilon}\right).$$

Proof. In order to prove Theorem 5.8 we have to examine how Lemmas 5.3, 5.5, 5.6, 5.7 and Proposition 5.4 can be adapted to the new regularity assumptions.

We begin with Lemma 5.3. Now we cannot apply Taylor expansion, since the solution is not regular enough. Thus we have to rewrite all the estimates. First we have

$$\begin{aligned} X(t^{n+1/2}) - (x - v\Delta t/2) &= X(t^{n+1/2}) - \left(X(t^{n+1}) - V(t^{n+1})\tfrac{\Delta t}{2}\right) \\ &= \int_{t^{n+1}}^{t^{n+1/2}} \left(V(t) - V(t^{n+1})\right) dt \\ &= \int_{t^{n+1}}^{t^{n+1/2}} \int_{t^{n+1}}^{t} E\left(\tau, X(\tau)\right) d\tau dt \\ &\le C\Delta t^2 \left\|E\right\|_{L^\infty(0,T;L^\infty([0,L]))} \\ &\le C\Delta t^2. \end{aligned} \tag{5.15}$$

Next we note that we have the following decomposition:

$$f(t^{n+1/2},y,v) - f\left(t^n, y - v\tfrac{\Delta t}{2}, v\right) = \int_{t^n}^{t^{n+1/2}} \partial_t f(t,y,v)dt + \int_{y-v\Delta t/2}^{y} \partial_x f(t^n,x,v)dx.$$

As $f \in \mathscr{C}_b(0,T;{W_c}^{1,\infty}(Q))$ and $\partial_t f \in L^\infty\left(0,T;L_c^\infty(Q)\right)$, integrating the previous decomposition, we obtain

$$\begin{aligned} &\int_0^L \int_{\mathbb{R}_v} \left| f(t^{n+1/2},y,v) - f\left(t^n, y - v\tfrac{\Delta t}{2}, v\right)\right| dydv \\ &\quad \le \int_0^L \int_{\mathbb{R}_v} \int_{t^n}^{t^{n+1/2}} \left|\partial_t f\left(t,y,v\right)\right| dtdvdy + \int_0^L \int_{\mathbb{R}_v} \int_{y-v\Delta t/2}^{y} \left|\partial_x f\left(t^n,x,v\right)\right| dxdvdy, \end{aligned}$$

and then

$$\int_0^L \int_{\mathbb{R}_v} \Big| f(t^{n+1/2}, y, v) - f\Big(t^n, y - v\tfrac{\Delta t}{2}, v\Big)\Big| dy dv$$
$$\leq CLQ^2(T)\Delta t \left(\|\partial_t f\|_{L^\infty(0,T,L^\infty(Q))} + \|\partial_x f\|_{L^\infty(0,T;L^\infty(Q))}\right)$$
$$\leq C\Delta t, \tag{5.16}$$

so that, using (2.7),

$$|E(t^{n+1/2}, x) - \widetilde{E}(t^{n+1/2}, x)| \leq C\Delta t. \tag{5.17}$$

Then we have

$$\begin{aligned} V(t^n) - \widetilde{V}(t^n) &= \int_{t^{n+1}}^{t^n} E\left(t, X(t)\right) dt + \Delta t \widetilde{E}\Big(t^{n+1/2}, x - v\tfrac{\Delta t}{2}\Big) \\ &= \int_{t^{n+1}}^{t^n} (E\left(t, X(t)\right) - E(t^{n+1/2}, X(t)))dt \\ &+ \int_{t^{n+1}}^{t^n} (E(t^{n+1/2}, X(t)) - E(t^{n+1/2}, X(t^{n+1/2})))dt \\ &+ \Delta t \Big\{ E\Big(t^{n+1/2}, X(t^{n+1}) - V(t^{n+1})\tfrac{\Delta t}{2}\Big) - E(t^{n+1/2}, X(t^{n+1/2}))\Big\} \\ &+ \Delta t \Big\{ \widetilde{E}\Big(t^{n+1/2}, X(t^{n+1}) - V(t^{n+1})\tfrac{\Delta t}{2}\Big) \\ &\qquad - E\Big(t^{n+1/2}, X(t^{n+1}) - V(t^{n+1})\tfrac{\Delta t}{2}\Big)\Big\}. \end{aligned}$$

As $E \in \mathscr{C}^1(0, T; \mathscr{C}^1_{b,per_x}(\mathbb{R}_x))$, we obtain

$$\begin{aligned} &\sup \Big\{ \Big| V(t^n; t^{n+1}, x, v) - \widetilde{V}(t^n; t^{n+1}, x, v)\Big| \mid \forall (x, v) \in [0, L] \times \mathbb{R}\Big\} \\ &\quad \leq C\Delta t^2 \mathrm{Lip}\left(E(., x)\right) + CQ(T)\Delta t^2 \mathrm{Lip}\left(E(t, .)\right) + C\Delta t^3 \mathrm{Lip}\left(E(t, .)\right) + C\Delta t^2 \\ &\quad \leq C\Delta t^2. \end{aligned} \tag{5.18}$$

We go on with the estimate of $X(t^n) - \widetilde{X}(t^n)$. We have

$$X(t^n) - \widetilde{X}(t^n) = \int_{t^{n+1}}^{t^n} \int_{t^{n+1}}^{t} E\left(\tau, X(\tau)\right) d\tau dt - \tfrac{\Delta t^2}{2} \widetilde{E}\Big(t^{n+1/2}, X(t^{n+1}) - V(t^{n+1})\tfrac{\Delta t}{2}\Big)$$

so that

$$\sup \Big\{ \Big| X(t^n) - \widetilde{X}(t^n)\Big| \mid \forall (x, v) \in [0, L] \times \mathbb{R}\Big\} \leq C\Delta t^2 \left(\|E\|_{L^\infty} + \|\widetilde{E}\|_{L^\infty}\right) \leq C\Delta t^2. \tag{5.19}$$

Now we use the estimates (5.18) and (5.19) in order to bound the quantity

$$\mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n) - f\left(t^{n+1}, x, v\right) = f(t^n, \widetilde{X}(t^n), \widetilde{V}(t^n)) - f\left(t^n, X(t^n), V(t^n)\right)$$

in the L^∞ norm. As we have the continuous embedding $W^{1,\infty} \hookrightarrow \mathscr{C}^{0,1}$, then

$$\begin{aligned} \left\|\mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n) - f\left(t^{n+1}\right)\right\|_{L^\infty(Q)} &\le C \mathrm{Lip}\left(f(t^n, ., .)\right) \Delta t^2, \\ &\le C \sup_{t \in [0,T]} \mathrm{Lip}\left(f(t, ., .)\right) \Delta t^2, \\ &\le C \Delta t^2. \end{aligned} \tag{5.20}$$

Following the proof of Proposition 5.4, if we take $f \in \mathscr{C}_b(0,T; W^{1,\infty}_c(Q))$, $E \in \mathscr{C}^1(0,T; \mathscr{C}^1_{b,per_x}(\mathbb{R}))$, and if we take the derivative in the sense of distribution, then, using (3.1), we still have (with $m \in \{0,1\}$)

$$||\mathcal{T}_i f||_{L^\infty(0,T;W^{m,\infty}(Q))} \le C ||f||_{L^\infty(0,T;W^{m,\infty}(Q))}, \tag{5.21}$$

$$||\widetilde{\mathcal{T}}_i f||_{L^\infty(0,T;L^\infty(Q))} \le C ||f||_{L^\infty(0,T;W^{m,\infty}(Q))}, \tag{5.22}$$

and

$$\|(\mathcal{T}_i - \widetilde{\mathcal{T}}_i) f\|_{L^\infty(0,T;L^\infty(Q))} \le C h ||f||_{L^\infty(0,T;W^{1,\infty}(Q))}. \tag{5.23}$$

As a consequence, Lemma 5.5 supplies the estimate

$$\|\mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n) - \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2 \circ \widetilde{\mathcal{T}}_1 f(t^n)\|_{L^\infty(Q)} \le C h.$$

The estimate of Lemma 5.6 has to be replaced by

$$\|\widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2 \circ \widetilde{\mathcal{T}}_1 f(t^n) - \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2^\star \circ \widetilde{\mathcal{T}}_1 f(t^n)\|_{L^\infty(Q)} \le C \Delta t \left(e^n + h\right).$$

In order to justify this inequality we just have to show that $\mathrm{Lip}(\widetilde{\mathcal{T}}_1 f(t^n))$ is bounded. Indeed we have

$$\mathrm{Lip}(\widetilde{\mathcal{T}}_1 f(t^n)) = \mathrm{Lip}\Big(\pi_h f\Big(t^n, x - v\tfrac{\Delta t}{2}, v\Big)\Big) \le \|\pi_h\|_{L^\infty} \mathrm{Lip}\left(f(t^n, ., .)\right) < +\infty.$$

Finally, we get all the desired a priori estimates by seeing that the stability result (5.13) still holds. Then the proof of the theorem is the same as that for Theorem 5.1, and we get

$$||f - f_h||_{\ell^\infty(0,T;L^\infty(Q))} \le C\left(\Delta t + h + \frac{h}{\Delta t}\right)$$

and

$$||E - E_h||_{\ell^\infty(0,T;L^\infty([0,L]))} \le C\left(\Delta t + h + \frac{h}{\Delta t}\right).$$

Now if we take $\Delta t \sim h^\varepsilon$ with $0 < \varepsilon < 1$, we get the desired result. In fact the best ε to choose is $1/2$ so that convergence holds with order $1/2$. $\square$

Remark 5.9. Under the regularity assumptions $f_0 \in \mathscr{C}^{m+1}_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)$, if there exists an interpolation operator π_h that satisfies both a consistency condition such as

$$\|f - \pi_h f\|_{L^\infty(0,T;L^p(Q))} \leq C h^{m+1} ||f||_{L^\infty(0,T;W^{m+1,p}(Q))} \tag{5.24}$$

and a stability condition such as

$$\|\pi_h f\|_{L^\infty(0,T;L^p(Q))} \leq (1 + Ch) ||f||_{L^\infty(0,T;L^p(Q))}, \tag{5.25}$$

then our method can easily be applied to prove the convergence of high order schemes in the L^p norm and to find error estimates such as

$$||f - f_h||_{\ell^\infty(0,T;L^p(Q))} \leq C\left(||f||_{\mathscr{C}^2(0,T;W^{m+1,p}(Q))}\right)\left(\Delta t^2 + h^{m+1} + \frac{h^{m+1}}{\Delta t}\right)$$

and

$$||E - E_h||_{\ell^\infty(0,T;L^\infty([0,L]))} \leq C\left(||f||_{\mathscr{C}^2(0,T;W^{m+1,p}(Q))}\right)\left(\Delta t^2 + h^{m+1} + \frac{h^{m+1}}{\Delta t}\right).$$

Unfortunately Lagrange interpolations of high order do not satisfy the stability condition (5.25). Besides, it seems difficult but not impossible to build interpolation operators π_h which satisfy both conditions (5.24) and (5.25).

If we use a Lagrange interpolation operator of high order, the discrete solution $f_h(t^n)$ belongs to $W^{1,p}(Q)$. The numerical scheme consists of a succession of transport and projection on the finite element space generated by the Lagrange finite element of high order. The transport operation leaves the norm of the solution unchanged. Then the scheme is stable if the interpolation operator π_h is stable, i.e., $\|\pi_h\|_{L^p} \leq 1 + \varepsilon(h)$ with $\lim_{h\to 0} \varepsilon(h) = 0$. Unfortunately Lagrange interpolation does not have nice properties of stability. Let $\tau_{h,\xi}$ be a translation operator such that $\tau_{z,\xi} f_h(t^n, x, v) = f_h(t^n, x - z, v - \xi) = g_h(t^n, x, v)$. Therefore $g_h(t^n) \in W^{1,p}(Q)$, and we have

$$\begin{aligned}
\|\pi_h \circ \tau_{z,\xi} f_h(t^n)\|_{L^p(Q)} &= \|\pi_h g_h(t^n)\|_{L^p(Q)} \\
&\leq \|g_h(t^n)\|_{L^p(Q)} + \|\pi_h g_h(t^n) - g_h(t^n)\|_{L^p(Q)} \\
&\leq \|g_h(t^n)\|_{L^p(Q)} + Ch\, |g_h(t^n)|_{W^{1,p}(Q)} \\
&\leq \|g_h(t^n)\|_{L^p(Q)} + C,
\end{aligned}$$

since $|g_h(t^n)|_{W^{1,p}(Q)} \sim O(h^{-1})$ and with C independent of h and such that $C > 1$.

We can also prove the convergence of our numerical scheme with noncompactly supported initial data. If we take f_0 such that

$$f_0 \in \mathscr{C}_{b,per_x} \cap W^{1,\infty} \cap W^{1,1}(\mathbb{R}_x \times \mathbb{R}_v),$$

$$0 < f_0 \leq (1 + |v|)^{-\lambda}, \quad v\nabla f_0 \in L^\infty_x\left(L^1_v\right),$$

and if we suppose that there exists a constant $R > 0$ such that

$$\mathcal{L}(f_0, R)(\xi) = \sup\left\{\frac{|f_0(x,v) - f_0(y,w)|}{\|(x,v) - (y,w)\|_2} \;\middle|\; \begin{array}{l} x, y \in [0,L],\ v, w \in \mathbb{R}, \\ (x,v) \neq (y,w),\ |v - \xi| \leq R,\ |w - \xi| \leq R \end{array}\right\}(1 + |\xi|) \in L^\infty \cap L^1(\mathbb{R}_\xi), \tag{5.26}$$

where $(x,v) \in [0,L] \times \mathbb{R}$ and $\|(x,v)\|_2 = \sqrt{x^2+v^2}$, the periodic Vlasov–Poisson system given by (2.7), (2.8), (2.9), and (2.10) has a unique solution (f,E) such that

$$0 < f(t,x,v) \le (1+|v|)^{-\lambda}, \tag{5.27}$$

$$f \in \mathscr{C}_b\left(0,T;\mathscr{C}_{b,per_x} \cap W^{1,\infty} \cap W^{1,1}(\mathbb{R}_x \times \mathbb{R}_v)\right),$$

$$v\nabla f,\ \partial_t f \in L^\infty\left(0,T;L^\infty_x\left(L^1_v\right)\right),$$

where the derivative is taken in the sense of distribution and

$$E \in \mathscr{C}^1\left(0,T;\mathscr{C}^1_{b,per_x}(\mathbb{R}_x)\right).$$

In addition, there exists a constant $C(T) > 0$ such that $\forall t \in [0,T]$,

$$\begin{aligned}&\mathcal{L}(f(t), R+C(T))(\xi)\\ &\quad = \sup\left\{ \frac{|f(t,x,v)-f(t,y,w)|}{\|(x,v)-(y,w)\|_2} \;\middle|\; x,y \in [0,L],\ v,w \in \times\mathbb{R},\right.\\ &\qquad\qquad \left.(x,v) \ne (y,w),\ |v-\xi| \le R+C(T),\ |w-\xi| \le R+C(T)\right\}(1+|\xi|)\\ &\quad \in L^\infty \cap L^1(\mathbb{R}_\xi).\end{aligned}$$

Now we state the theorem.

THEOREM 5.10. *Assume that* $f_0 \in \mathscr{C}_{b,per_x} \cap W^{1,\infty} \cap W^{1,1}(\mathbb{R}_x \times \mathbb{R}_v)$, $0 \le f_0 \le (1+|v|)^{-\lambda}$, $\forall \lambda > 1$ *and that* f_0 *satisfies* (5.26). *Let* α *be such that* $0 < \alpha < \lambda$, *and suppose that the bound of velocity support* R *evolves as* $h^{-1/\alpha}$. *Then* (f_h, E_h), *the numerical solution of the periodic Vlasov–Poisson system, converges towards* (f,E), *and there exists a positive function* μ *such that* $\lim_{h\to 0}\mu(h) = 0$, *and a constant* $C = C(\|f\|_{L^\infty(0,T;W^{1,\infty}(Q))}, \|f\|_{L^\infty(0,T;W^{1,1}(Q))}, \|\partial_t f\|_{L^\infty(0,T;L^\infty_x(L^1_v))}, \|v\nabla f\|_{L^\infty(0,T;L^\infty_x(L^1_v))})$ *independent of* Δt, h *such that*

$$||f - f_h||_{\ell^\infty(0,T;L^{1,\infty}(Q))} \le C\left(\Delta t + h + (h+\mu(h))^{1-1/\sigma} + h^{\lambda/\alpha}\right)$$

and

$$||E - E_h||_{\ell^\infty(0,T;L^\infty([0,L]))} \le C\left(\Delta t + h + (h+\mu(h))^{1-1/\sigma} + h^{\lambda/\alpha}\right),$$

where $\Delta t \sim (h+\mu(h))^{1/\sigma}$, *with* $\sigma > 1$.

Before giving the proof of Theorem 5.10, we need to establish the L^1 stability of π_h by proving the following two lemmas.

LEMMA 5.11. *Assume that* $0 \le f_0(x,v) \le \zeta(x,v) \sim (1+|v|)^{-\lambda}$, *for* $\lambda > 1$. *Then there exists a constant* C, *depending only on* T, L, *and* f_0, *such that*

$$0 \le f_h(t,x,v) \le C\zeta_h(x,v), \quad t \in [0,T], \quad (x,v) \in Q, \tag{5.28}$$

where

$$\zeta_h(x,v) = \sum_k \frac{1}{(1+|v_k|)^\lambda}\psi_k(x,v).$$

There also exists a constant $C > 0$ *such that*

$$||f_h(t)||_{L^1(Q)} \le C, \quad t \in [0,T]. \tag{5.29}$$

Proof. We begin with the transport in x. Let us notice that there exists a constant R independent of h such that for every triangle T_m of the triangulation $\mathcal{T}_h$ there exists a ball $B(a_m, Rh)$ of center a_m and radius Rh which contains T_m. Let N_k be a vertex of triangle T_m. If we consider transport in x, the origin of the characteristic, $x_k^\star = x_k - v_k\Delta t/2$, which ends at N_k, belongs to a triangle $T_m^\star$. Let $\mu(h)$ be a positive function such that $\lim_{h\to 0}\mu(h) = 0$. If N_o, N_p, and N_q are the vertices of the triangle $T_m^\star$, we have

$$|v_k - v_o| \le 2Rh \le 2R(h + \mu(h)) \le 2R\varepsilon(h),$$

$$|v_k - v_p| \le 2Rh \le 2R(h + \mu(h)) \le 2R\varepsilon(h),$$

and

$$|v_k - v_q| \le 2Rh \le 2R(h + \mu(h)) \le 2R\varepsilon(h),$$

where $\varepsilon(h) = h + \mu(h)$. On the other hand, we note that

$$\frac{\zeta_h(x_j, v_j)}{\zeta_h(x_k, v_k)} = \frac{(1 + |v_k|)^\lambda}{(1 + |v_j|)^\lambda} \le 1 + C_1(\lambda, R)\varepsilon(h), \quad j = \{o, p, q\}. \tag{5.30}$$

Now, if we consider the transport in v, the origin of the characteristic $v_k^* = v_k - E_h(t^{n+1/2})\Delta t$ which ends at N_k belongs to a triangle T_m^*. If N_i, N_s, and N_l are the vertices of a triangle T_m^*, as $\|E_h\|_{\ell^\infty(0,T;L^\infty([0,L]))}$ is bounded we have

$$|v_k - v_i| \le C\Delta t, \quad |v_k - v_s| \le C\Delta t, \quad \text{and } |v_k - v_l| \le C\Delta t,$$

and then we have

$$\frac{\zeta_h(x_j, v_j)}{\zeta_h(x_k, v_k)} = \frac{(1 + |v_k|)^\lambda}{(1 + |v_j|)^\lambda} \le 1 + C_2(\lambda, R)\Delta t, \quad j = \{i, s, l\}. \tag{5.31}$$

If we set $b_1 = 1 + C_1(\lambda, R)\varepsilon(h)$, $b_2 = 1 + C_2(\lambda, R)\Delta t$, and $b = b_1 b_2 b_1$, then we have

$$f_h(0, x_k, v_k) \le \frac{1}{(1 + |v_k|)^\lambda} \le \frac{b^0}{(1 + |v_k|)^\lambda}$$

and consequently

$$f_h(0, x, v) \le b^0 \zeta_h(x, v).$$

If we assume that

$$f_h(t^n, x_k, v_k) \le b^n \zeta_h(x_k, v_k)$$

and consequently

$$f_h(t^n, x, v) \le b^n \zeta_h(x, v),$$

the numerical scheme gives for the first half advection with the respect to the variable x

$$f_h(t^{n+1/2}, x_k, v_k) = \sum_l f_h(t^n, x_l, v_l)\psi_l\left(x_k - v_k\frac{\Delta t}{2}, v_k\right).$$

Let $T_m^\star$ be the triangle which contains the origin of the characteristic coming from the node N_k. Let N_o, N_p, and N_q be the three vertices of $T_m^\star$. Then we can write

$$\frac{f_h(t^{n+1/2}, x_k, v_k)}{\zeta_h(x_k, v_k)} = \lambda_o\frac{f_{h,o}^n}{\zeta_h(x_k, v_k)} + \lambda_p\frac{f_{h,p}^n}{\zeta_h(x_k, v_k)} + \lambda_q\frac{f_{h,q}^n}{\zeta_h(x_k, v_k)},$$

where

$$\lambda_l = \psi_l\left(x_k - v_k\frac{\Delta t}{2}, v_k\right)$$

and

$$f_{h,l}^n = f_h(t^n, x_l, v_l).$$

Using the property (5.30) of ζ_h and the property

$$\lambda_o + \lambda_p + \lambda_q = 1,$$

we obtain

$$\frac{f_h(t^{n+1/2}, x_k, v_k)}{\zeta_h(x_k, v_k)} \leq b^n\lambda_o\frac{\zeta_h(x_o, v_o)}{\zeta_h(x_k, v_k)} + b^n\lambda_p\frac{\zeta_h(x_p, v_p)}{\zeta_h(x_k, v_k)} + b^n\lambda_q\frac{\zeta_h(x_q, v_q)}{\zeta_h(x_k, v_k)} \leq b_1 b^n.$$

In the same way for the two other advections we finally obtain

$$\frac{f_h(t^{n+1}, x_k, v_k)}{\zeta_h(x_k, v_k)} \leq b^{(n+1)} \quad \forall\ N_k \in \mathcal{T}_h.$$

For a finite time T and $\forall n \in \{0, \ldots, T/\Delta t\}$, if we consider $\varepsilon(h) \leq \Delta t$, we have $b \leq 1 + C(C_1, C_2)\Delta t$, $b^{(n+1)} \leq \exp(C(C_1, C_2)T)$, and as in the continuous case there exists a majorizing function of the discrete distribution

$$f_h(t, x, v) \leq C\zeta_h(x, v) \quad \forall t \in [0, T], \quad \forall (x, v) \in Q.$$

In order to prove (5.29), we note that

$$\begin{aligned}\int_{\mathbb{R}\zeta_h(x,v)dxdv} &= \sum_k \frac{1}{(1 + |v_k|)^\lambda}\int_{\mathbb{R}}\psi_k(x, v)dxdv \\ &= \sum_k \frac{\text{meas}(\mathcal{A}_k)}{(1 + |v_k|)^\lambda} \\ &\leq C\int_{\mathbb{R}}\frac{1}{(1 + |v|)^\lambda} < +\infty. \qquad \square\end{aligned}$$

Let $\mathcal{A}_k$ be the area associated with the node N_k and $\psi_k \in P_1$; then we have

$$\text{meas}(\mathcal{A}_k) = \int_{\mathbb{R}}\psi_k(x, v)dxdv = \frac{|\text{supp}\psi_k|}{3}.$$

We introduce χ_k, the characteristic function defined as follows:

$$\chi_k(x,v) = \begin{cases} 1 \text{ if } (x,v) \in \mathcal{A}_k, \\ 0 \text{ otherwise.} \end{cases}$$

Then we introduce the function $g_h^n(x,v)$ defined by

$$g_h^n(x,v) = \sum_k g_h^n(x_k,v_k)\chi_k(x,v),$$

with

$$g_h^n(x_k,v_k) = f_h(t^n,x_k,v_k).$$

We note that

$$||f_h(t^n)||_{L^1(Q)} = ||g_h^n||_{L^1(Q)}.$$

Moreover, as for the proof of the Lemma 5.11, we can prove that

$$0 \le g_h^n(x,v) \le C\gamma_h(x,v) \ \ \forall n \in [0,N], \ N = \left[\frac{T}{\Delta t}\right], \quad (x,v) \in Q,$$

where

$$\gamma_h(x,v) = \sum_k \frac{1}{(1+|v_k|)^\lambda}\chi_k(x,v).$$

We notice that there exists another constant C independent of h such that

$$0 < \gamma_h \le C(1+|v|)^{-\lambda}.$$

Now we state the lemma which shows the L^1 stability of the interpolation operator π_h.

LEMMA 5.12. *Let $g \in \mathscr{C}_b \cap L^1(Q)$ and $0 < g \le C(1+|v|)^{-\lambda}$; then there exists a positive function μ, where $\lim_{h\to 0}\mu(h) = 0$, such that*

$$||\pi_h g||_{L^1(Q)} \le ||g||_{L^1(Q)} + \mu(h).$$

Proof. We have

$$\begin{aligned} ||\pi_h g||_{L^1(Q)} &= \sum_k g_k \int_{\mathbb{R}}\int_0^L \psi_k(x,v)dxdv = \sum_k g_k \mathrm{meas}(\mathcal{A}_k) \\ &= \sum_k g_k \int_Q \chi_k(x,v)dxdv = \int_Q \sum_k g_k\chi_k(x,v)dxdv \\ &= ||g_h||_{L^1(Q)}. \end{aligned}$$

As

$$g_h(x,v) \le C(1+|v|)^{-\lambda}$$

and

$$\lim_{h\to 0} g_h = g \quad \text{a.e.},$$

the dominated convergence theorem asserts that

$$\lim_{h\to 0} \int_Q |g_h - g|\, dxdv = 0,$$

and as a consequence there exists a positive function μ with $\lim_{h\to 0}\mu(h) = 0$ such that

$$\int_Q |g_h - g|\, dxdv \le \mu(h).$$

Then we deduce that

$$\left|\|\pi_h g\|_{L^1(Q)} - \|g\|_{L^1(Q)}\right| = \left|\|g_h\|_{L^1(Q)} - \|g\|_{L^1(Q)}\right| \le \int_Q |g_h - g|\, dxdv \le \mu(h).$$

Finally we deduce that

$$||\pi_h g||_{L^1(Q)} \le ||g||_{L^1(Q)} + \mu(h). \qquad \square$$

Now we can return to the proof of Theorem 5.10.

Proof of Theorem 5.10. In order to prove the theorem we have to see how the a priori estimates (5.16), (5.17), (5.18), (5.19), and (5.20) are modified and obtain the same kind of a priori estimates in the L^1 norm.

As $v\nabla f,\ \partial_t f \in L^\infty(0,T; L^\infty_x(L^1_v))$ the estimate (5.16) becomes

$$\begin{aligned}\int_0^L \int_{\mathbb{R}_v} &|f(t^{n+1/2}, y, v) - f\left(t^n, y - v\Delta t/2, v\right)| dydv \\ &\le CL\Delta t \left(\|\partial_t f\|_{L^\infty(0,T,L^\infty_x(L^1_v))} + \|v\partial_x f\|_{L^\infty(0,T;L^\infty_x(L^1_v))}\right) \\ &\le C\Delta t,\end{aligned}$$

so that we still have

$$|E(t^{n+1/2}, x) - \widetilde{E}(t^{n+1/2}, x)| \le C\Delta t.$$

The estimate (5.19) still holds, but the estimate (5.18) changes into

$$|V(t^n) - \widetilde{V}(t^n)| \le C(1 + |v|)\Delta t^2.$$

Then the estimate (5.20) becomes

$$\begin{aligned}\left\|\mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n) - f\left(t^{n+1}\right)\right\|_{L^\infty(Q)} &\le \sup_{v\in\mathbb{R}} \left\{\mathcal{L}(f(t^n), C(T))(v)\right\} \Delta t^2 \\ &\le C\Delta t^2,\end{aligned}$$

and in the L^1 norm we have

$$
\begin{aligned}
&\left\|\mathcal{T}_1\circ\mathcal{T}_2\circ\mathcal{T}_1 f(t^n) - f\left(t^{n+1}\right)\right\|_{L^1(Q)}\\
&\leq \int_0^L\int_{\mathbb{R}}\Big| f\left(t^n,\widetilde{X}(t^n;t^{n+1},x,v),\widetilde{V}(t^n;t^{n+1},x,v)\right)\\
&\qquad\qquad -f\left(t^n,X(t^n;t^{n+1},x,v),V(t^n;t^{n+1},x,v)\right)\Big| dvdx\\
&\leq \int_0^L\int \sup\Big\{\|(\chi,\xi)-(y,w)\|_2^{-1}\cdot|f(t^n,\chi,\xi)-f(t^n,y,w)|\ \Big|\\
&\qquad\qquad (\chi,\xi),(y,w)\in[0,L]\times\mathbb{R},\ (\chi,\xi)\neq(y,w),|\xi-v|,|w-v|\leq C(T)\Big\}\\
&\qquad\qquad \times\left\|\left(\widetilde{X}(t^n;t^{n+1},x,v),\widetilde{V}(t^n;t^{n+1},x,v)\right)\right.\\
&\qquad\qquad\quad \left.-\left(X(t^n;t^{n+1},x,v),V(t^n;t^{n+1},x,v)\right)\right\|_2 dvdx\\
&\leq \Delta t^2\int_0^L\int_{\mathbb{R}}\mathcal{L}(f(t^n),C(T))(v)dvdx\\
&\leq C\Delta t^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\sup\left\{\left|V(t^n;t^{n+1},x,v)-v\right|\ |\ x\in[0,L],v\in\mathbb{R}\right\} &\leq \int_{t^n}^{t^{n+1}}\|E(\tau,.)\|_{L^\infty}\,dt\\
&\leq T\,\|E\|_{L^\infty(0,T;L^\infty)}\leq C(T)<+\infty.
\end{aligned}
$$

Then we conclude that the estimate of Lemma 5.3 has to be replaced by

$$\left\|\mathcal{T}_1\circ\mathcal{T}_2\circ\mathcal{T}_1 f(t^n)-f\left(t^{n+1}\right)\right\|_{L^{1,\infty}(Q)}\leq C\Delta t^2.$$

Following the proof of Proposition 5.4, if we take $f\in\mathscr{C}_b(0,T;W^{1,\infty}\cap W^{1,1}(Q))$ and $E\in\mathscr{C}^1(0,T;\mathscr{C}^1_{b,per_x}(\mathbb{R}))$, then using (3.1) and taking the derivative in the sense of distribution, we still have (with $m\in\{0,1\}$, $p\in\{1,\infty\}$)

$$||\mathcal{T}_i f||_{L^\infty(0,T;W^{m,p}(Q))}\leq C||f||_{L^\infty(0,T;W^{m,p}(Q))},$$

$$||\widetilde{\mathcal{T}}_i f||_{L^\infty(0,T;L^p(Q))}\leq C||f||_{L^\infty(0,T;W^{m,p}(Q))},$$

and

$$\|(\mathcal{T}_i-\widetilde{\mathcal{T}}_i)f\|_{L^\infty(0,T;L^p(Q))}\leq Ch||f||_{L^\infty(0,T;W^{1,p}(Q))}.$$

As a consequence, Lemma 5.5 supplies the estimate

$$\|\mathcal{T}_1\circ\mathcal{T}_2\circ\mathcal{T}_1 f(t^n)-\widetilde{\mathcal{T}}_1\circ\widetilde{\mathcal{T}}_2\circ\widetilde{\mathcal{T}}_1 f(t^n)\|_{L^{1,\infty}(Q)}\leq Ch.$$

The estimate of Lemma 5.6 has to be replaced by

$$\|\widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2 \circ \widetilde{\mathcal{T}}_1 f(t^n) - \widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2^\star \circ \widetilde{\mathcal{T}}_1 f(t^n)\|_{L^{1,\infty}(Q)} \leq C\Delta t \left(e^n + \frac{1}{(1+R)^\lambda} + h \right), \tag{5.32}$$

where

$$e^n = \|f(t^n) - f_h(t^n)\|_{L^{1,\infty}(Q)}\,.$$

The proof of Lemma 5.6 holds, except for the estimate of $E_h^{n+1/2}(x) - \widetilde{E}^{n+1/2}(x)$ that we slightly modify as follows. We rewrite

$$\begin{aligned}
&E_h^{n+1/2}(x) - \widetilde{E}^{n+1/2}(x) \\
&\quad = \int_0^L K(x,y) \left(\int_{\mathbb{R}} \left[\widetilde{\mathcal{T}}_1 f_h(t^n, y, v) - \mathcal{T}_1 f(t^n, y, v) \right] dv \right) dy \\
&\quad = \int_0^L K(x,y) \left(\int_{|v|\leq R} \pi_h \Big[f_h\Big(t^n, y - v\tfrac{\Delta t}{2}, v\Big) - f\Big(t^n, y - v\tfrac{\Delta t}{2}, v\Big) \Big] dv \right) dy \\
&\qquad + \int_0^L \int_{|v|>R} K(x,y) f\Big(t^n, y - v\tfrac{\Delta t}{2}, v\Big) dv dy \\
&\qquad + \int_0^L \int_{|v|\leq R} K(x,y) \Big(\pi_h f\Big(t^n, y - v\tfrac{\Delta t}{2}, v\Big) - f\Big(t^n, y - v\tfrac{\Delta t}{2}, v\Big) \Big) dv dy,
\end{aligned}$$

so that we get

$$\begin{aligned}
\|E_h^{n+1/2} - \widetilde{E}^{n+1/2}\|_{L^\infty([0,L])} \leq{} & ||K||_{L^\infty} ||\pi_h||_{L^\infty} \, \|f_h(t^n) - f(t^n)\|_{L^{1,\infty}(Q)} \\
& + ||K||_{L^\infty} ||f(t^n)||_{L^1(Q\setminus\Omega)} \\
& + C||K||_{L^\infty} h ||f(t^n)||_{W^{1,1}(Q)}.
\end{aligned} \tag{5.33}$$

Thanks to assumption (5.27), for the second term of (5.33) we obtain

$$\left\| E_h^{n+1/2} - \widetilde{E}^{n+1/2} \right\|_{L^\infty([0,L])} \leq C \left(e^n + \frac{1}{(1+R)^\lambda} + h \right).$$

In order to finish justifying the inequality (5.32), we now just have to show that $\mathcal{L}(\widetilde{\mathcal{T}}_1 f(t^n, C(T)))(\xi)$ belongs to $L^\infty \cap L^1$. Indeed we have

$$\begin{aligned}
\mathcal{L}(\widetilde{\mathcal{T}}_1 f(t^n, C(T)))(\xi) &= \mathcal{L}\Big(\pi_h f\Big(t^n, x - v\tfrac{\Delta t}{2}, v\Big), C(T)\Big)(\xi) \\
&\leq \|\pi_h\|_{L^\infty} \mathcal{L}\left(f(t^n), C(T)\right)(\xi) \in L^\infty \cap L^1.
\end{aligned}$$

Finally, thanks to Lemma 5.12, we get the $L^{1,\infty}$ stability of the interpolation operator π_h; that is to say, there exists a constant C such that

$$||\pi_h f||_{L^{1,\infty}} \leq ||f||_{L^{1,\infty}} + \mu(h) \quad \forall f \in \mathscr{C}_b\left(0, T; \mathscr{C}_{b,per_x} \cap L^1(\mathbb{R}_x \times \mathbb{R}_v)\right).$$

Then it is obvious that the estimate of Lemma 5.7 becomes

$$\|\widetilde{\mathcal{T}}_1 \circ \widetilde{\mathcal{T}}_2^\star \circ \widetilde{\mathcal{T}}_1 (f(t^n) - f_h(t^n))\|_{L^{1,\infty}(Q)} \leq e^n + 3\mu(h).$$

As in the proof of the main theorem, a discrete Gronwall inequality enables us to get

$$e^{n+1} \le \exp(CT)e^0 + C\left(\Delta t + h + \frac{h+\mu(h)}{\Delta t} + \frac{1}{(1+R)^\lambda}\right).$$

If we suppose that $R = \frac{1}{h^{1/\alpha}}$, $\alpha > 0$, and since e^0 is only a fixed interpolation error, we obtain

$$e^{n+1} \le C\left(\Delta t + h + \frac{h+\mu(h)}{\Delta t} + h^{\lambda/\alpha}\right).$$

Then the end of the proof is the same as the proof of the main Theorem 5.1, and we get

$$||f - f_h||_{\ell^\infty(0,T;L^{1,\infty}(Q))} \le C\left(\Delta t + h + \frac{h+\mu(h)}{\Delta t} + h^{\lambda/\alpha}\right)$$

and

$$||E - E_h||_{\ell^\infty(0,T;L^\infty([0,L]))} \le C\left(\Delta t + h + \frac{h+\mu(h)}{\Delta t} + h^{\lambda/\alpha}\right).$$

If we choose $\Delta t \sim (h+\mu(h))^{1/\sigma}$, $\sigma > 1$, we get the estimates of Theorem 5.1. ∎

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