Mathematical Models and Methods in Applied Sciences
Vol. 22, No. 1 (2012) 1150001 (43 pages)

DOI: 10.1142/S0218202512005848

GLOBAL WEAK SOLUTIONS FOR THE RELATIVISTIC WATERBAG CONTINUUM

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Received 9 September 2010
Revised 11 March 2011
Communicated by F. Bouchut

In this paper we consider the relativistic waterbag continuum which is a useful PDE for collisionless kinetic plasma modeling recently developed in Ref. 11. The waterbag representation of the statistical distribution function of particles can be viewed as a special class of exact weak solution of the Vlasov equation, allowing to reduce this latter into a set of hydrodynamic equations (with the complexity of a multi-fluid model) while keeping its kinetic features (Landau damping and nonlinear resonant wave-particle interaction). These models are very promising because they are very useful for analytical theory and numerical simulations of laser-plasma and gyrokinetic physics.[10–16, 56, 57] The relativistic waterbag continuum is derived from two phase-space variable reductions of the relativistic Vlasov–Maxwell equations through the existence of two underlying exact invariants, one coming from physics properties of the dynamics is the canonical transverse momentum, and the second, named the "water-bag" and coming from geometric property of the phase-space is just the direct consequence of the Liouville Theorem. In this paper we prove the existence and uniqueness of global weak entropy solutions of the relativistic waterbag continuum. Existence is based on vanishing viscosity method and bounded variations (BV) estimates to get compactness while proof of uniqueness relies on kinetic formulation of the relativistic waterbag continuum and the associated kinetic entropy defect measure.

Keywords: Waterbag model; relativistic Vlasov–Maxwell equations; hyperbolic conservation laws; entropic weak solutions; kinetic formulation; plasma physics.

AMS Subject Classification: 35A01, 35A02, 35A05, 35D05, 35D10, 35D30, 35D40, 76X05, 82D10

1. Introduction

Vlasov equation is a difficult one mainly because of its high dimensionality. For each particle species, the distribution function $f(t, \mathbf{r}, \mathbf{v})$ is defined in a 6D phase-space. Even the simplest (one spatial dimension, one velocity dimension) implies a 2D phase-space. Can it be reduced to the sole configuration space as in usual hydrodynamics? In that last case the presence of collisions with frequency much greater than the inverse of all characteristic times implies the existence of a local thermodynamic equilibrium characterized by a density $n(t, \mathbf{r})$, an average velocity $\mathbf{u}(t, \mathbf{r})$ and a temperature $T(t, \mathbf{r})$. *A priori* in a plasma the distribution function $f(t, \mathbf{r}, \mathbf{v})$ is an arbitrary function of $\mathbf{r}$ and $\mathbf{v}$ (and t of course) and phase-space is unavoidable.

An alternative approach is based on a waterbag representation of the distribution function which is not an approximation but rather a special class of initial conditions. Introduced initially by DePackh,[26] Hohl, Feix and Bertrand[8,9,32,33] the waterbag model was shown to bring the bridge between fluid and kinetic description of a collisionless plasma, allowing to keep the kinetic aspect of the problem (such as Landau damping and nonlinear resonant wave-particle interaction) with the same complexity as a multi-fluid model. Twenty years later, mathematicians have rediscovered this property using the kinetic formulation of scalar conservation laws. It was established in Refs. 19, 20 and 36 that scalar conservation laws can be lifted as linear hyperbolic equations by introducing an extra variable $\xi \in \mathbb{R}$ which can be interpreted as a scalar momentum or velocity variable. In Ref. 20 the author proposed a numerical scheme, known as the transport-collapse method to solve this linear kinetic equation and has proved, using BV estimates and Kruzhkov type analysis, that this numerical solution converges to the entropy solution of scalar conservation laws. This result was also shown in Ref. 64 using averaging lemmas[17,29,39,40] without bounded variation estimates. Soon after, it was shown in Refs. 61, 54 and 59 that, without any approximations, entropy solutions of scalar conservation laws could be directly formulated in kinetic style, known as kinetic formulation. Its generalization to systems of conservation laws seems impossible except for very peculiar systems[5–7,21,55,65] where among others, the kinetic formulation of multibranch entropy solutions has been developed. One of those systems is the isentropic gas dynamics system with $\gamma = 3$ for which, long time ago, the link with the Vlasov kinetic equation was pointed out in Ref. 8 as the so-called waterbag model. Let us notice that the multibranch entropy solutions have been used for multivalued geometric optics computations and multiphase computations of the semiclassical limit of the Schrödinger equation.[41–43,46]

This paper deals with the electromagnetic relativistic waterbag continuum which arises from the reduction of four-dimensional relativistic Vlasov–Maxwell equations. Usefulness and efficiency of this model have been recently shown in the context of laser-plasma interaction[11] where it has been used to recover Landau damping effect,[51,58] Van Kampen modes propagation,[23,63] nonlinear Bohm–Gross frequency shift of plasma waves[9,27] and treat stimulated Raman scattering instability at the saturation regime.[14,30,31,34,44] Moreover, this model reveals to be very useful and powerful to explain the formation of stable coherent low-frequency nonlinear structures such as KEEN (kinetic electron electrostatic nonlinear) and EAW (electron acoustic-like) waves which appear in laser-plasma interaction at nonlinear stage and persist in the long time dynamics. These modes which have been observed in several simulations[1,2,14,35] can be viewed as a non-steady variant of the well-known Bernstein–Greene–Kruskal[4] (BGK) modes that describe invariant traveling electrostatic waves in plasmas.[45,49,50] The ability of the waterbag model to supply a scenario for the formation of coherent low-frequency structures is very promising and advanced research on this topic is under consideration. Let us notice that the application of the waterbag model in magnetic controlled fusion, where plasma gyrokinetic turbulence governs the energy confinement time, has provided satisfying and hopeful results.[10–13,15,16,56,57]

The paper is organized as follows. Section 2 deals with the derivation of the relativistic waterbag continuum from a four-dimensional relativistic Vlasov–Maxwell system. Section 3 is devoted to the proof of the existence and uniqueness of global weak solutions of the relativistic waterbag continuum. Existence is based on the vanishing viscosity method which leads to the existence of strong solution sequences of a regularized problem. Therefore, *a priori* bounded variation (BV) estimates are recovered to obtain compactness of the solution sequences and pass to the limit in the weak (in the distributional sense) formulation of the problem. Uniqueness of weak solutions relies on kinetic formulation of the relativistic waterbag continuum and the associated kinetic entropy defect measure which are equivalent to the weak entropy solution notion. Finally, the kinetic formulation allows to make the link between the weak entropy solution of the relativistic waterbag continuum, and a special class of weak solutions of the relativistic Vlasov–Maxwell equations with kinetic entropy defect measure.

2. The Relativistic Waterbag Continuum

Since we want to describe the behavior of an electromagnetic wave propagation in a relativistic gas with a fixed neutralizing ion background, we need to solve the relativistic Vlasov–Maxwell equations. Even for a plasma plane wave propagating, let us say along the x-direction, we have to solve a Vlasov equation for a four-dimensional distribution function $F = F(t, x, p_x, \mathbf{p}_\perp)$ with $\mathbf{p}_\perp = (p_y, p_z)$:

$$\partial_t F + \frac{p_x}{m\gamma}\partial_x F + q\left(\mathbf{E} + \frac{\mathbf{p} \times \mathbf{B}}{m\gamma}\right) \cdot \nabla_{\mathbf{p}} F = 0, \tag{2.1}$$

where m is the mass of the particles and $q = Ze$ is the signed electrical charge with Z the number of charge and e the signed elementary electrical charge. The

Lorentz factor γ reads $\gamma^2 = 1 + |\mathbf{p}|^2/(m^2c^2)$ where $|\cdot|$ denotes the Euclidean norm. Furthermore, it is easy to reduce the four-dimensional relativistic Vlasov–Maxwell equations (2.1) into two-dimensional Vlasov equation in the following way. Let us consider the Hamiltonian of a particle in the electromagnetic field $(\mathbf{E}, \mathbf{B})$, in the relativistic regime, for a one-dimensional system (x) (i.e. for the plane wave propagation), $\mathscr{H} = mc^2(\sqrt{1 + |\mathbf{P}_c - q\mathbf{A}|^2/(m^2c^2)} - 1) + q\phi$ (rest mass energy, mc^2, has been dropped has it remains constant and plays no role here), where $\phi = \phi(t, x)$ is the electrostatic potential, $\mathbf{A} = \mathbf{A}(t, x)$ is the vector potential and $\mathbf{P}_c$ is the canonical momentum connected to the particle momentum $\mathbf{p}$ by $\mathbf{P}_c = \mathbf{p} + q\mathbf{A}$. Choosing the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) implies that $\mathbf{A} = \mathbf{A}_\perp(t, x)$. If we write Hamiltonian equation $\frac{d\mathbf{P}_c}{dt} = -\nabla_{\mathbf{q}}\mathscr{H}$, where $\mathbf{q} = (x, y, z)$, then along the longitudinal x-direction of propagation of the electromagnetic wave we have $\frac{dP_{cx}}{dt} = -\partial_x\mathscr{H}$, and for the transverse (y, z)-direction $\frac{d\mathbf{P}_{c\perp}}{dt} = -\nabla_{\mathbf{q}_\perp}\mathscr{H} = 0$. The last equation means $\mathbf{P}_{c\perp} = \text{constant} = \mathscr{P}_{c\perp}$ and $\mathbf{P}_{c\perp}$ is no more an independent variable but a parameter or a label. Therefore we can consider solution for the Vlasov equation (2.1) of the form

$$F(t, x, p_x, \mathbf{p}_\perp) = \int_{\mathbb{R}^2} f(t, x, p_x, \mathscr{P}_{c\perp})\delta(\mathbf{p}_\perp - \mathbf{P}_\perp(t, x, \mathscr{P}_{c\perp}))d\nu(\mathscr{P}_{c\perp}),$$

where $\mathbf{P}_\perp(t, x, \mathscr{P}_{c\perp}) = \mathscr{P}_{c\perp} - q\mathbf{A}_\perp$, and $\nu(d\mathscr{P}_{c\perp})$ denotes an absolutely continuous measure with respect to the Lebesgue measure. The Hamiltonian of one particle of transverse canonical momentum invariant $\mathscr{P}_{c\perp}$ is given by $\mathscr{H}_r = \mathscr{H}_r(t, x, p_x, \mathscr{P}_{c\perp}) = mc^2(\gamma_r(t, x, p_x, \mathscr{P}_{c\perp}) - 1) + q\phi$, where $\gamma_r(t, x, p_x, \mathscr{P}_{c\perp})^2 = 1 + (p_x^2 + |\mathbf{P}_\perp(t, x, \mathscr{P}_{c\perp})|^2)/(m^2c^2)$. Each set of particles of transverse canonical momentum invariant $\mathscr{P}_{c\perp}$, is described by a distribution function $f(t, x, p_x, \mathscr{P}_{c\perp})$ which obeys the Vlasov equation: $\partial_t f + [\mathscr{H}_r, f] = 0$, for all $\mathscr{P}_{c\perp} \in \mathbb{R}^2$, where $[\varphi, \psi] = \partial_{p_x}\phi\partial_x\psi - \partial_x\varphi\partial_{p_x}\psi$. Therefore the two-dimensional Vlasov equation reads

$$\partial_t f + \frac{p_x}{m\gamma_r}\partial_x f + \left(qE_x - \frac{1}{2m\gamma_r}\partial_x|\mathbf{P}_{c\perp}|^2\right)\partial_{p_x} f = 0, \quad \forall\, \mathscr{P}_{c\perp} \in \mathbb{R}^2. \tag{2.2}$$

If we now consider two Lagrangian foliations to be the families of sheets $p^\pm(t, x, a, \mathscr{P}_{c\perp})$, labeled by the Lagrangian label $a \in [0, 1]$, where the waterbag continuum $p^\pm(t, x, a, \mathscr{P}_{c\perp})$ are smooth functions, we define the distribution function $f(t, x, p_x, \mathscr{P}_{c\perp})$ such that

$$f(t, x, p_x, \mathscr{P}_{c\perp}) = \int_0^1 d\mu(a)(\mathrm{H}(p^+(t, x, a, \mathscr{P}_{c\perp}) - p_x) - \mathrm{H}(p^-(t, x, a, \mathscr{P}_{c\perp}) - p_x)), \tag{2.3}$$

where $\mu(da)$ denotes an absolutely continuous positive measure with respect to the Lebesgue measure and H is the Heaviside function. Therefore the waterbag distribution function (2.3) is a solution of the Vlasov equation (2.2), in the distributional sense if and only if

$$\partial_t p^\pm + \frac{p^\pm}{m\gamma_r^\pm}\partial_x p^\pm - \left(qE_x - \frac{1}{2m\gamma_r^\pm}\partial_x|\mathbf{P}_{c\perp}|^2\right) = 0, \quad \mathscr{P}_{c\perp} \in \mathbb{R}^2, \quad a \in [0,1], \tag{2.4}$$

where $\gamma_r^\pm = \gamma_r(t, x, p^\pm, \mathscr{P}_{c\perp})$. The last Eq. (2.4) can be rewritten in the conservative form

$$\partial_t p^\pm + \partial_x \mathcal{H}^\pm = 0, \quad \mathscr{P}_{c\perp} \in \mathbb{R}^2, \quad a \in [0,1], \tag{2.5}$$

where $\mathcal{H}^\pm = \mathcal{H}(t, x, p^\pm, \mathscr{P}_{c\perp}) = mc^2(\gamma_r^\pm - 1) + q\phi$. The canonical transverse momentum $\mathscr{P}_{c\perp}$ is an exact physical invariant while the bag $(p^+ - p^-)da$ is an exact geometric invariant, which is reminiscent to the geometric Liouville invariant.

We now add the Maxwell equations which couple the waterbag continuum $p^\pm$, through the scalar potential ϕ and the vector potential $\mathbf{A}_\perp$. The one-dimensional wave-propagation model allows one to separate the electric field into two parts, namely $\mathbf{E} = E_x\mathbf{e}_x + \mathbf{E}_\perp = -\nabla\phi - \partial_t\mathbf{A}$, where $E_x = -\partial_x\phi$ is a pure electrostatic field which obeys Poisson equation, and $\mathbf{E}_\perp = -\partial_t\mathbf{A}_\perp$ is a pure electromagnetic field. In the absence of any external magnetic field, $\mathbf{B}$ is purely perpendicular and is given by $\mathbf{B}_\perp = \nabla \times \mathbf{A}_\perp$. The other two Maxwell equations $\partial_t\mathbf{B} + \nabla \times \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$ are automatically satisfied. The two others couple the waterbag continuum $p^\pm$. The Maxwell–Gauss equation becomes

$$-\partial_x^2\phi = \frac{q}{\varepsilon_0}(\rho - n_0), \quad E_x = -\partial_x\phi, \tag{2.6}$$

where

$$\begin{aligned}\rho &= \int_{\mathbb{R}^3} F(t, x, p_x, \mathbf{p}_\perp)d\mathbf{p}_\perp dp_x = \int_{\mathbb{R}^3} f(t, x, p_x, \mathscr{P}_{c\perp})dp_x d\nu(\mathscr{P}_{c\perp}),\\ &= \int_0^1 d\mu(a)\int_{\mathbb{R}^2} d\nu(\mathscr{P}_{c\perp})\int_{p^-}^{p^+} dp_x,\\ &= \int_0^1 d\mu(a)\int_{\mathbb{R}^2} d\nu(\mathscr{P}_{c\perp})\{p^+(t, x, a, \mathscr{P}_{c\perp}) - p^-(t, x, a, \mathscr{P}_{c\perp})\}.\end{aligned}$$

Let us notice that we can equivalently replace the Poisson equation by the longitudinal x-component of the Maxwell–Ampère equation to compute the longitudinal electric field E_x

$$\partial_t E_x = -\frac{1}{\varepsilon_0}J_x, \tag{2.7}$$

where

$$
\begin{aligned}
J_x &= \frac{q}{m}\int_{\mathbb{R}^3} \frac{p_x}{\gamma} F(t,x,p_x,\mathbf{p}_\perp) d\mathbf{p}_\perp dp_x,\\
&= \frac{q}{m}\int_{\mathbb{R}^3} \frac{p_x}{\gamma_r(p_x,\mathbf{P}_\perp(t,x,\mathscr{P}_{c\perp}))} f(t,x,p_x,\mathscr{P}_{c\perp}) dp_x d\nu(\mathscr{P}_{c\perp}),\\
&= \frac{q}{m}\int_0^1 d\mu(a)\int_{\mathbb{R}^2} d\nu(\mathscr{P}_{c\perp})\int_{p^-}^{p^+} \frac{p_x}{\gamma_r(p_x,\mathbf{P}_\perp(t,x,\mathscr{P}_{c\perp}))} dp_x,\\
&= \frac{q}{m}\int_0^1 d\mu(a)\int_{\mathbb{R}^2} d\nu(\mathscr{P}_{c\perp})\{\mathcal{H}(t,x,p^+(t,x,a,\mathscr{P}_{c\perp}),\mathscr{P}_{c\perp})\\
&\quad - \mathcal{H}(t,x,p^-(t,x,a,\mathscr{P}_{c\perp}),\mathscr{P}_{c\perp})\}.
\end{aligned}
$$

The Maxwell–Ampère equation $\nabla \times \mathbf{B}_\perp = \mu_0(\mathbf{J}_\perp + \varepsilon_0 \partial_t \mathbf{E}_\perp)$ and the Maxwell–Faraday equation $\partial_t \mathbf{B}_\perp + \nabla \times \mathbf{E}_\perp = 0$ can be combined to get the waves equation

$$
\partial_t^2 \mathbf{A}_\perp - c^2 \partial_x^2 \mathbf{A}_\perp = \mu_0 \mathbf{J}_\perp, \tag{2.8}
$$

where

$$
\begin{aligned}
\mathbf{J}_\perp &= \frac{q}{m}\int_{\mathbb{R}^3} \frac{\mathbf{p}_\perp}{\gamma} F(t,x,p_x,\mathbf{p}_\perp) d\mathbf{p}_\perp dp_x,\\
&= \frac{q}{m}\int_{\mathbb{R}^3} \frac{\mathbf{P}_\perp(t,x,\mathscr{P}_{c\perp})}{\gamma_r(p_x,\mathbf{P}_\perp(t,x,\mathscr{P}_{c\perp}))} f(t,x,p_x,\mathscr{P}_{c\perp}) dp_x d\nu(\mathscr{P}_{c\perp}),\\
&= \frac{q}{m}\int_{\mathbb{R}^2} d\nu(\mathscr{P}_{c\perp})\mathbf{P}_\perp(t,x,\mathscr{P}_{c\perp})\int_0^1 d\mu(a)\int_{p^-(t,x,a,\mathscr{P}_{c\perp})}^{p^+(t,x,a,\mathscr{P}_{c\perp})} \frac{dp_x}{\gamma_r(p_x,\mathbf{P}_\perp(t,x,\mathscr{P}_{c\perp}))}.
\end{aligned}
$$

If we now consider the canonical transverse momemtum measure $\nu(d\mathscr{P}_{c\perp}) = \sum_{j=1}^{\mathcal{M}} \delta(\mathscr{P}_{c\perp} - \mathscr{P}_{c\perp,j}) d\mathscr{P}_{c\perp}$, then it means that the plasma is initially well prepared so that particles are divided in $\mathcal{M}$ bunches of particles, each bunch j having the same initial perpendicular canonical momentum $\mathscr{P}_{c\perp,j}$. Choosing $\mu(da) = \sum_{i=1}^{\mathcal{N}} \delta(a - a_i) da$, means that the waterbag continuum is in fact a multiple waterbag.[11] In the sequel, without loss of generality from the analysis point of view, we choose $\nu(d\mathscr{P}_{c\perp}) = \delta(\mathscr{P}_{c\perp}) d\mathscr{P}_{c\perp}$, which means that we consider a cold plasma with no streaming effect in the transverse direction. Moreover, without loss of generality we assume that the measure $\mu(da)$ is absolutely continuous with uniform density equal to one, i.e. $d\mu(a) = da$ where da denotes the Lebesgue measure. Moreover, as it is commonly done in plasma physics, we assume that the position space is the one-dimensional torus of sidelength L, namely $\mathbb{T}_L = \mathbb{R}/(L\mathbb{Z})$. We also assume that $p \in \mathbb{R}$ and $a \in [0,1]$. We note $\mathcal{D} = \mathbb{T}_L \times [0,1]$, $Q = [0,T] \times \mathcal{D}$, $\Omega = [0,T] \times \mathbb{T}_L$, $\mathfrak{D} = \mathbb{T}_L \times \mathbb{R}$, $\mathfrak{Q} = [0,T] \times \mathfrak{D}$ and $\Sigma = [0,T] \times \mathcal{D} \times \mathbb{R}$. Since the problem is posed on the one-dimensional torus in space, the Maxwell–Ampère equation (2.7) should be modified by adding on the right-hand side of Eq. (2.7) the current average over one period L. To complete the system, we need to add the initial conditions $p^{\pm}(t=0,x,a) = p_0^{\pm}(x,a)$, $\mathbf{A}_{\perp}(t=0,x) = \mathbf{A}_{\perp}^0(x)$ and $\partial_t \mathbf{A}_{\perp}(t=0,x) = \mathbf{A}_{\perp}^1(x)$. After classical normalization, the dimensionless relativistic waterbag continuum $p^{\pm} = p^{\pm}(t,x,a)$ and the electromagnetic field $(\phi, \mathbf{A}_{\perp})$ satisfy the readily obtained relativistic waterbag equations (RWB)

$$\partial_t p^{\pm} + \partial_x \mathcal{H}(t,x,p^{\pm}) = 0, \tag{2.9}$$

$$\partial_t^2 \mathbf{A}_{\perp} - \partial_x^2 \mathbf{A}_{\perp} = -\mathbf{A}_{\perp}\rho_{\gamma}, \tag{2.10}$$

$$E_x = -\partial_x \phi, \quad -\partial_x^2 \phi = \rho - 1, \tag{2.11}$$

$$\partial_t E_x = -J_x + \frac{1}{L}\int_{\mathbb{T}_L} J_x(t,x)dx, \tag{2.12}$$

where

$$\begin{aligned}\mathcal{H}(t,x,p) &= \gamma(t,x,p) - 1 + \phi(t,x)\\ &= \sqrt{1+p^2+|\mathbf{A}_{\perp}(t,x)|^2} + \phi(t,x) - 1,\end{aligned} \tag{2.13}$$

$$\rho_{\gamma}(t,x) = \int_0^1 da \int_{p^-(t,x,a)}^{p^+(t,x,a)} \frac{dp}{\gamma(t,x,p)}, \tag{2.14}$$

$$\rho(t,x) = \int_0^1 da(p^+(t,x,a) - p^-(t,x,a)), \tag{2.15}$$

$$\begin{aligned}J_x(t,x) &= \int_0^1 da \int_{p^-(t,x,a)}^{p^+(t,x,a)} \frac{pdp}{\gamma(t,x,p)}\\ &= \int_0^1 da\{\mathcal{H}(t,x,p^+(t,x,a)) - \mathcal{H}(t,x,p^-(t,x,a))\},\end{aligned} \tag{2.16}$$

with the initial conditions $p^{\pm}(t=0,\cdot,\cdot) = p_0^{\pm}(\cdot,\cdot)$, $\mathbf{A}_{\perp}(t=0,\cdot) = \mathbf{A}_{\perp}^0(\cdot)$, and $\partial_t \mathbf{A}_{\perp}(t=0,\cdot) = \mathbf{A}_{\perp}^1(\cdot)$.

3. Existence and Uniqueness of Global Weak Solutions

In this section we will show existence and uniqueness of global weak solutions of the relativistic waterbag continuum (2.9)–(2.12). To achieve this aim, we first show the global existence of strong solutions of a regularized problem. Afterwards, we establish *a priori* bounded variation estimates on the solution sequences which allow to pass weakly to the limit in the regularized problem to obtain weak solutions of the system (2.9)–(2.12). Therefore we can state some remarkable properties of the obtained solutions such as the preservation of the order. These properties give a supplementary information on the structure of the solution, especially on the monotonicity of the waterbag continuum with respect to the a-variable. Finally we use kinetic formulation of the system (2.9)–(2.12), which is equivalent to the weak entropy solutions notion, in order to establish a L^1-stability property of the solutions with respect to their initial data and show uniqueness of the global weak solutions. We complete the study by showing the link between the weak entropy solutions of the relativistic waterbag continuum that we have obtained and special class of weak solutions of the relativistic Vlasov–Maxwell equations with kinetic entropy defect measure.

3.1. *Global existence for a regularized problem*

We first introduce a regularization of the system (2.9)–(2.12) by substituting to Eq. (2.9) its viscous regularization

$$\partial_t p^{\pm} + \partial_x \mathcal{H}(t, x, p^{\pm}) = \varepsilon \partial_x^2 p^{\pm}, \tag{3.1}$$

where the parameter $\varepsilon > 0$ stands for a viscosity. Before going further, let us define some functional spaces. Let us first define the vector-valued Lebesgue space $\mathbb{L}^2$ such as $\mathbb{L}^2 = L^2 \times L^2$. We next introduce the Hilbert space

$$V = \{\varphi \in L^2(\mathcal{D}) \,|\, \partial_x \varphi \in L^2(\mathcal{D})\} = L^2([0,1]; H^1(\mathbb{T}_L)), \tag{3.2}$$

equipped with the scalar product

$$\langle \varphi, \psi \rangle_V = \int_0^1 \langle \varphi, \psi \rangle_{H^1(\mathbb{T}_L)} da = \sum_{\alpha=0}^{1} \int_0^1 \langle \partial_x^\alpha \varphi, \partial_x^\alpha \psi \rangle_{L^2(\mathbb{T}_L)} da,$$

and the norm $\|\varphi\|_V = \sqrt{\langle \varphi, \varphi \rangle_V}$. The dual space of V is the Banach space V' defined by[52]

$$V' = L^2([0,1]; H^{-1}(\mathbb{T}_L)). \tag{3.3}$$

Let us note that $V \subset L^2(\mathcal{D}) \subset V'$ and that V is dense in $L^2(\mathcal{D})$. Therefore we can introduce the space $\mathcal{W}(0,T)$ defined as

$$\mathcal{W}(0,T) = \{\varphi \in L^2(0,T;V); \partial_t \varphi \in L^2(0,T;V')\}. \tag{3.4}$$

Using the definition (3.4) of the space $\mathcal{W}(0,T)$, we have the following theorem.

Theorem 3.1. (Global strong solutions) *Let us assume that* $p_0^{\pm} \in L^2(\mathcal{D})$, $\mathbf{A}_\perp^0 \in \mathbb{L}^2(\mathbb{T}_L)$ *and* $\mathbf{A}_\perp^1 \in \mathbb{L}^2(\mathbb{T}_L)$, *then the system* (3.1) *and* (2.10)–(2.12) *has a unique global strong solution*

$$\begin{cases} p^{\pm} \in \mathscr{C}(0,T; L^2(\mathcal{D})) \cap \mathcal{W}(0,T), \\ \phi \in \mathscr{C}(0,T; L^2(\mathbb{T}_L)) \quad \textit{and} \quad \mathbf{A}_\perp \in \mathscr{C}(0,T; \mathbb{L}^2(\mathbb{T}_L)). \end{cases} \tag{3.5}$$

The proof of Theorem 3.1 relies on a change of unknowns related to the waterbag continuum, classical results concerning linear parabolic equations,[52] the Banach fixed point theorem, energy estimates and Gronwall lemma. Even if the proof of Theorem 3.1 is rather classical, for the sake of completeness, it is outlined in Appendix A. We next establish a global strong existence result when $p_0^\pm \in L^2 \cap L^\infty(\mathcal{D})$.

Theorem 3.2. (Global strong solutions) *Let us assume that* $p_0^\pm \in L^2 \cap L^\infty(\mathcal{D})$, *and* $\mathbf{A}_\perp^0$, $\mathbf{A}_\perp^0{}'$, $\mathbf{A}_\perp^1 \in \mathbb{L}^2 \cap \mathbb{L}^\infty(\mathbb{T}_L)$, *then the system* (3.1) *and* (2.10)–(2.12) *has a unique global strong solution*

$$p^\pm \in \mathscr{C}(0,T;L^2(\mathcal{D})) \cap \mathcal{W}(0,T) \cap L^\infty(Q),$$
$$\phi \in \mathscr{C}(0,T;L^2(\mathbb{T}_L)) \cap W^{1,\infty}(\Omega) \quad \textit{and} \quad \mathbf{A}_\perp \in \mathscr{C}(0,T;\mathbb{L}^2(\mathbb{T}_L)) \cap \mathbb{W}^{1,\infty}(\Omega).$$

Moreover, we have

$$\|p^\pm(t)\|_{L^\infty(\mathcal{D})} \le \|p_0^\pm\|_{L^\infty(\mathcal{D})} + \int_0^t d\tau \{\|\partial_x \phi(\tau)\|_{L^\infty(\mathbb{T}_L)} + \|\partial_x \mathbf{A}_\perp(\tau)\|_{\mathbb{L}^\infty(\mathbb{T}_L)}\}. \tag{3.6}$$

In Theorem 3.2 the prime notation in the expression $\mathbf{A}_\perp^0{}'$ stands for the partial derivative with respect to the x-variable. The proof of Theorem 3.2 relies on the truncation method of Stampacchia,[22, 38] energy estimates and Gronwall lemma. Although quite classical, the proof of Theorem 3.2 is outlined in Appendix B for the sake of completeness.

3.2. A priori *bounded variation estimates*

In order to pass to the limit as the viscosity ε tends to zero in the weak formulation (in sense of distribution or in $\mathscr{D}'$) of the regularized problem we need to obtain *a priori* bounded variation estimates independent of the parameter ε which will lead to the compactness property of the solution sequences. We establish the following theorem

Theorem 3.3. (BV estimates) *Let us assume that* $p_0^\pm \in L^2 \cap L^\infty \cap \mathrm{BV}(\mathcal{D})$, *and* $\mathbf{A}_\perp^0$, $\mathbf{A}_\perp^0{}'$, $\mathbf{A}_\perp^1 \in \mathbb{L}^2 \cap \mathbb{L}^\infty \cap \mathbb{BV}(\mathbb{T}_L)$, *then the system* (3.1) *and* (2.10)–(2.12) *has a unique global strong solution*

$$p^\pm \in \mathscr{C}(0,T;L^2(\mathcal{D})) \cap \mathcal{W}(0,T) \cap L^\infty(Q) \cap L^\infty(0,T;\mathrm{BV}(\mathcal{D})),$$
$$\phi \in \mathscr{C}(0,T;L^2(\mathbb{T}_L)) \cap W^{1,\infty}(\Omega) \quad \textit{and} \quad \mathbf{A}_\perp \in \mathscr{C}(0,T;\mathbb{L}^2(\mathbb{T}_L)) \cap \mathbb{W}^{1,\infty}(\Omega).$$

Moreover, there exists a constant $\mathcal{C}_{\mathrm{BV}}$ *independent of* ε, *but which may depend on* $\|p_0^\pm\|_{\mathrm{BV}(\mathcal{D})}$, $\|p^\pm\|_{L^\infty(Q)}$, $\|\partial_x \mathbf{A}_\perp\|_{\mathbb{L}^\infty(\Omega)}$, $\|\mathbf{A}_\perp^0\|_{\mathbb{BV}(\mathbb{T}_L)}$, $\|\mathbf{A}_\perp^0{}'\|_{\mathbb{BV}(\mathbb{T}_L)}$, *and* $\|\mathbf{A}_\perp^1\|_{\mathbb{BV}(\mathbb{T}_L)}$ *such that*

$$\|p^\pm\|_{L^\infty(0,T;\mathrm{BV}(\mathcal{D}))} \le \mathcal{C}_{\mathrm{BV}}.$$

Proof. Let $\zeta_h \in \mathscr{C}_0^\infty(\mathbb{R})$ be a convex regularization of the modulus function which converges uniformly to $|\cdot|$ as $h \to 0$ and satisfies $|\zeta_h'| \le 1$. Let $(p_i^\pm, \phi_i :=$

$\phi[\mathbf{p}_i], \mathbf{A}_{\perp,i} := \mathbf{A}_\perp[\mathbf{p}_i])$, with $i = 1, 2$, be two solutions of the system (3.1) and (2.10)–(2.12) with initial conditions $(p^\pm_{0\,i}, \mathbf{A}^0_{\perp,i}, \mathbf{A}^1_{\perp,i})$ for $i = 1, 2$. The notation $\phi[\mathbf{p}]$ (respectively, $\mathbf{A}_\perp[\mathbf{p}]$) with $\mathbf{p} = (p^-, p^+)$, means that the electrical potential ϕ (respectively, vector potential $\mathbf{A}_\perp$) depends on $\mathbf{p}$ through the source term of the Poisson (respectively, Ampère) equation. In other words, this notation is used to stress the fact that the electrical potential ϕ (respectively, vector potential $\mathbf{A}_\perp$) has to be seen as an integral operator acting on $\mathbf{p}$ or a map from $\mathbf{p} \in L^\infty(0, T; \mathbb{L}^2(\mathcal{D}))$ into $L^\infty(0, T; L^2(\mathcal{D}))$ (respectively, $L^\infty(0, T; \mathbb{L}^2(\mathcal{D}))$). We then set $p^\pm = p^\pm_1 - p^\pm_2$, $\phi = \phi_1 - \phi_2$ and $\mathbf{A}_\perp = \mathbf{A}_{\perp,1} - \mathbf{A}_{\perp,2}$. If we multiply Eq. (3.1) by $\zeta'_h(p^\pm)$, after integration we obtain

$$\frac{d}{dt}\int_0^1 da \int_{\mathbb{T}_L} dx\, \zeta_h(p^\pm) + \int_0^1 da \int_{\mathbb{T}_L} dx\, \zeta'_h(p^\pm)(\partial_x\gamma^\pm(\mathbf{p}_1) - \partial_x\gamma^\pm(\mathbf{p}_2))$$
$$= -\int_0^1 da \int_{\mathbb{T}_L} dx\, \zeta'_h(p^\pm)\partial_x\phi[\mathbf{p}] - \varepsilon\int_0^1 da \int_{\mathbb{T}_L} dx\, \zeta''_h(p^\pm)|\partial_x p^\pm|^2. \qquad (3.7)$$

Since the ζ_h is convex, the second term of the right-hand side of (3.7) is nonpositive. Since the operator $\partial_x\phi[\cdot] : L^1(\mathbb{T}_\mathbb{L}) \to L^1(\mathbb{T}_\mathbb{L})$ is bounded in L^1 and $|\zeta'_h| \le 1$, we have

$$\left|\int_0^1 da \int_{\mathbb{T}_L} dx\, \zeta'_h(p^\pm)\partial_x\phi[\mathbf{p}]\right|$$
$$\le \|K\|_{L^1(\mathbb{T}_L)\times L^\infty(\mathbb{T}_L)}(\|p^-_1 - p^-_2\|_{L^1(\mathcal{D})} + \|p^+_1 - p^+_2\|_{L^1(\mathcal{D})}), \qquad (3.8)$$

where $K(x, y) = -\partial_x G(x, y)$ and $G \in W^{1,\infty}(\mathbb{T}^2_L)$ is the Green function of the one-dimensional Laplace operator with periodic boundary conditions. Using integrations by parts, and the identity

$$\zeta''_h(p^\pm)p^\pm\partial_x p^\pm = \partial_x\int_0^{p^\pm} \zeta''_h(s)s ds,$$

we obtain

$$\int_0^1 da \int_{\mathbb{T}_L} dx\, \zeta'_h(p^\pm)(\partial_x\gamma^\pm(\mathbf{p}_1) - \partial_x\gamma^\pm(\mathbf{p}_2))$$
$$= \int_0^1 da \int_{\mathbb{T}_L} dx\, \partial_x\left(\frac{\gamma^\pm(\mathbf{p}_1) - \gamma^\pm(\mathbf{p}_2)}{p^\pm}\right)\int_0^{p^\pm} \zeta''_h(s)s ds$$
$$= \int_0^1 da \int_{\mathbb{T}_L} dx\, \partial_x\left(\frac{p^\pm_1 + p^\pm_2}{\gamma^\pm(\mathbf{p}_1) + \gamma^\pm(\mathbf{p}_2)}\right)\int_0^{p^\pm} \zeta''_h(s)s ds$$
$$+ \int_0^1 da \int_{\mathbb{T}_L} dx\, \partial_x\left(\frac{|\mathbf{A}_{\perp,1}|^2 - |\mathbf{A}_{\perp,2}|^2}{p^\pm(\gamma^\pm(\mathbf{p}_1) + \gamma^\pm(\mathbf{p}_2))}\right)\int_0^{p^\pm} \zeta''_h(s)s ds. \qquad (3.9)$$

Let us first estimate the first term of the right-hand side of (3.9). After expanding the x-derivative and using obvious estimates we obtain

$$\int_0^1 da \int_{\mathbb{T}_L} dx\, \partial_x \left(\frac{p_1^\pm + p_2^\pm}{\gamma^\pm(\mathbf{p}_1) + \gamma^\pm(\mathbf{p}_2)} \right) \int_0^{p^\pm} \zeta_h''(s) s ds$$
$$\leq \int_0^1 da \int_{\mathbb{T}_L} dx\, \{2|\partial_x p_1^\pm| + 2|\partial_x p_2^\pm| + |\partial_x \mathbf{A}_{\perp,1}| + |\partial_x \mathbf{A}_{\perp,2}|\} \int_0^{p^\pm} \zeta_h''(s) s ds$$
$$\leq \epsilon(h)(\|p_1^\pm\|_{\mathrm{BV}(\mathcal{D})} + \|p_2^\pm\|_{\mathrm{BV}(\mathcal{D})} + \|\partial_x \mathbf{A}_{\perp,1}\|_{\mathbb{L}^1(\mathbb{T}_L)} + \|\partial_x \mathbf{A}_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}), \tag{3.10}$$

where

$$\epsilon(h) = C \sup_{p\in\mathbb{R}} \left| \int_0^p \zeta_h''(s) s ds \right| \xrightarrow{h\to 0} 0 \quad \text{uniformly}.$$

In Eq. (3.10), we have used the fact that, for $i = 1, 2$, $\|\partial_x p_i^\pm\|_{L^1(\mathcal{D})} = \|p_i^\pm\|_{L^1(0,1;\mathrm{BV}(\mathbb{T}_L))} \leq \|p_i^\pm\|_{\mathrm{BV}(\mathcal{D})}$, since for a.e. $t \in [0, T]$, $p_i^\pm(t) \in V$ (see definition (3.2)). For the second term of the right-hand side of (3.9), after integrating by parts, expanding the x-derivative, using obvious estimates and $|\zeta_h'| \leq 1$ we obtain

$$\int_0^1 da \int_{\mathbb{T}_L} dx\, \partial_x \left(\frac{|\mathbf{A}_{\perp,1}|^2 - |\mathbf{A}_{\perp,2}|^2}{p^\pm(\gamma^\pm(\mathbf{p}_1) + \gamma^\pm(\mathbf{p}_2))} \right) \int_0^{p^\pm} \zeta_h''(s) s ds$$
$$= \int_0^1 da \int_{\mathbb{T}_L} dx\, \partial_x \left(\frac{|\mathbf{A}_{\perp,1}|^2 - |\mathbf{A}_{\perp,2}|^2}{\gamma^\pm(\mathbf{p}_1) + \gamma^\pm(\mathbf{p}_2)} \right) \zeta_h'(p^\pm) ds$$
$$\leq 3\,(\|\partial_x \mathbf{A}_{\perp,1}\|_{\mathbb{L}^\infty(\mathbb{T}_L)} + \|\partial_x \mathbf{A}_{\perp,2}\|_{\mathbb{L}^\infty(\mathbb{T}_L)})\|\mathbf{A}_{\perp,1} - \mathbf{A}_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}$$
$$+ 2\|\partial_x \mathbf{A}_{\perp,1} - \partial_x \mathbf{A}_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}$$
$$+ \|\mathbf{A}_{\perp,1} - \mathbf{A}_{\perp,2}\|_{\mathbb{L}^\infty(\mathbb{T}_L)}(\|p_1^\pm\|_{\mathrm{BV}(\mathcal{D})} + \|p_2^\pm\|_{\mathrm{BV}(\mathcal{D})}). \tag{3.11}$$

Now using d'Alembert integral representation formula, we obtain

$$\|\mathbf{A}_{\perp,1} - \mathbf{A}_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}$$
$$\leq \|\mathbf{A}^0_{\perp,1} - \mathbf{A}^0_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)} + t\|\mathbf{A}^1_{\perp,1} - \mathbf{A}^1_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}$$
$$+ 2(\|p_1^-\|_{L^\infty(Q)} + \|p_1^+\|_{L^\infty(Q)})t \int_0^t d\tau\, \|\mathbf{A}_{\perp,1} - \mathbf{A}_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}$$
$$+ t \int_0^t d\tau\, (\|p_1^- - p_2^-\|_{L^1(\mathcal{D})} + \|p_1^+ - p_2^+\|_{L^1(\mathcal{D})}) \tag{3.12}$$

and

$$\|\partial_x \mathbf{A}_{\perp,1} - \partial_x \mathbf{A}_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}$$
$$\leq \|\mathbf{A}^{0\prime}_{\perp,1} - \mathbf{A}^{0\prime}_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)} + \|\mathbf{A}^1_{\perp,1} - \mathbf{A}^1_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}$$

$$+\left(\|p_1^-\|_{L^\infty(Q)}+\|p_1^+\|_{L^\infty(Q)}\right)\int_0^t d\tau\,\|\mathbf{A}_{\perp,1}-\mathbf{A}_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}$$
$$+\int_0^t d\tau\left(\|p_1^- - p_2^-\|_{L^1(\mathcal{D})}+\|p_1^+ - p_2^+\|_{L^1(\mathcal{D})}\right). \tag{3.13}$$

Using a Gronwall lemma, Eq. (3.12) leads to

$$\|\mathbf{A}_{\perp,1}-\mathbf{A}_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}$$
$$\le e^{2t^2(\|p_1^-\|_{L^\infty(Q)}+\|p_1^+\|_{L^\infty(Q)})}\Bigg(\|\mathbf{A}^0_{\perp,1}-\mathbf{A}^0_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}$$
$$+t\|\mathbf{A}^1_{\perp,1}-\mathbf{A}^1_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}+t\int_0^t d\tau\left(\|p_1^- - p_2^-\|_{L^1(\mathcal{D})}+\|p_1^+ - p_2^+\|_{L^1(\mathcal{D})}\right)\Bigg). \tag{3.14}$$

Substituting estimates (3.8)–(3.14) into the addition of Eqs. (3.7), after time integration, we get

$$\int_{\mathcal{D}} dxda\,\zeta_h(p_1^- - p_2^-)+\int_{\mathcal{D}} dxda\,\zeta_h(p_1^+ - p_2^+)$$
$$\le \int_{\mathcal{D}} dxda\,\zeta_h(p_{0\,1}^- - p_{0\,2}^-)+\int_{\mathcal{D}} dxda\,\zeta_h(p_{0\,1}^+ - p_{0\,2}^+)$$
$$+\widetilde{\mathcal{C}}\Bigg\{\|\mathbf{A}^0_{\perp,1}-\mathbf{A}^0_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}+\|\mathbf{A}^{0\prime}_{\perp,1}-\mathbf{A}^{0\prime}_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}+\|\mathbf{A}^1_{\perp,1}-\mathbf{A}^1_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}$$
$$+\int_0^t d\tau(\|p_1^- - p_2^-\|_{L^1(\mathcal{D})}+\|p_1^+ - p_2^+\|_{L^1(\mathcal{D})})\Bigg\}+\widehat{\mathcal{C}}\int_0^t d\tau\,\epsilon(h)$$
$$+\mathcal{C}^\natural\int_0^t d\tau\left(\|\mathbf{A}_{\perp,1}-\mathbf{A}_{\perp,2}\|_{\mathbb{L}^\infty(\mathbb{T}_L)}+\epsilon(h)\right)$$
$$\cdot\left(\|p_1^-\|_{\mathrm{BV}(\mathcal{D})}+\|p_2^-\|_{\mathrm{BV}(\mathcal{D})}+\|p_1^+\|_{\mathrm{BV}(\mathcal{D})}+\|p_2^+\|_{\mathrm{BV}(\mathcal{D})}\right), \tag{3.15}$$

where $\mathcal{C}^\natural$ is a purely numerical constant,

$$\widetilde{\mathcal{C}} := \widetilde{\mathcal{C}}(t,\|p_1^\pm\|_{L^\infty(Q)},\|\partial_x\mathbf{A}_{\perp,1}\|_{\mathbb{L}^\infty(\Omega)},\|\partial_x\mathbf{A}_{\perp,2}\|_{\mathbb{L}^\infty(\Omega)}),$$
$$\widehat{\mathcal{C}} := \widehat{\mathcal{C}}(\|\partial_x\mathbf{A}_{\perp,1}\|_{L^\infty(0,T;\mathbb{L}^1(\mathbb{T}_L))},\|\partial_x\mathbf{A}_{\perp,2}\|_{L^\infty(0,T;\mathbb{L}^1(\mathbb{T}_L))}),$$

and $\widetilde{\mathcal{C}}$ is nondecreasing in time. Let us introduce the translation operator τ_z defined as follows. For any $z=(z_x,z_a)\in\mathbb{R}^2$, for any functions $f\in \mathrm{BV}(\mathcal{D})$ and $g\in\mathrm{BV}(\mathbb{T}_L)$ we define $\tau_z f := f(\cdot+z_x,\cdot+z_a)$ and $\tau_z g := g(\cdot+z_x)$. Now we set $p_1^\pm = p^\pm$, $p_2^\pm=\tau_z p^\pm$, $p_{0\,1}^\pm = p_0^\pm$, $p_{0\,2}^\pm=\tau_z p_0^\pm$, $\mathbf{A}_{\perp,1}=\mathbf{A}_\perp$, $\mathbf{A}_{\perp,2}=\tau_z\mathbf{A}_\perp$, $\mathbf{A}^0_{\perp,1}=\mathbf{A}^0_\perp$, $\mathbf{A}^0_{\perp,2}=\tau_z\mathbf{A}^0_\perp$, $\mathbf{A}^1_{\perp,1}=\mathbf{A}^1_\perp$, and $\mathbf{A}^1_{\perp,2}=\tau_z\mathbf{A}^1_\perp$. Dividing inequality (3.15) by $|z|$, passing to the limit in inequality (3.15) as $h \to 0$ and using a Gronwall lemma we obtain

$$\begin{aligned}
&\frac{\|\tau_z p^- - p^-\|_{L^1(\mathcal{D})}}{|z|} + \frac{\|\tau_z p^+ - p^+\|_{L^1(\mathcal{D})}}{|z|} \\
&\quad \le e^{\widetilde{\mathcal{C}}t}\left\{ \frac{\|\tau_z p_0^- - p_0^-\|_{L^1(\mathcal{D})}}{|z|} + \frac{\|\tau_z p_0^+ - p_0^+\|_{L^1(\mathcal{D})}}{|z|} \right. \\
&\qquad + \widetilde{\mathcal{C}}\left[\frac{\|\tau_z \mathbf{A}_\perp^0 - \mathbf{A}_\perp^0\|_{\mathbb{L}^1(\mathbb{T}_L)}}{|z|} + \frac{\|\tau_z \mathbf{A}_\perp^0{}' - \mathbf{A}_\perp^0{}'\|_{\mathbb{L}^1(\mathbb{T}_L)}}{|z|} + \frac{\|\tau_z \mathbf{A}_\perp^1 - \mathbf{A}_\perp^1\|_{\mathbb{L}^1(\mathbb{T}_L)}}{|z|} \right] \\
&\qquad \left. + \mathcal{C}^\natural \int_0^t d\tau \, \frac{\|\tau_z \mathbf{A}_\perp - \mathbf{A}_\perp\|_{\mathbb{L}^\infty(\mathbb{T}_L)}}{|z|} (\|p^-\|_{\mathrm{BV}(\mathcal{D})} + \|p^+\|_{\mathrm{BV}(\mathcal{D})}) \right\}.
\end{aligned} \tag{3.16}$$

After taking the supremum in $z \in \mathbb{R}^2$ in (3.16) we obtain

$$\begin{aligned}
&\|p^-\|_{\mathrm{BV}(\mathcal{D})} + \|p^+\|_{\mathrm{BV}(\mathcal{D})} \\
&\quad \le e^{\widetilde{\mathcal{C}}t}\Big\{ \|p_0^-\|_{\mathrm{BV}(\mathcal{D})} + \|p_0^+\|_{\mathrm{BV}(\mathcal{D})} + \widetilde{\mathcal{C}}[\|\mathbf{A}_\perp^0\|_{\mathbb{BV}(\mathbb{T}_L)} + \|\mathbf{A}_\perp^0{}'\|_{\mathbb{BV}(\mathbb{T}_L)} \\
&\qquad + \|\mathbf{A}_\perp^1\|_{\mathbb{BV}(\mathbb{T}_L)}] + \mathcal{C}^\natural \|\partial_x \mathbf{A}_\perp\|_{\mathbb{L}^\infty(\Omega)} \int_0^t d\tau \, (\|p^-\|_{\mathrm{BV}(\mathcal{D})} + \|p^+\|_{\mathrm{BV}(\mathcal{D})}) \Big\}.
\end{aligned} \tag{3.17}$$

Finally using again a Gronwall lemma, estimate (3.17) leads

$$\begin{aligned}
&\|p^-\|_{L^\infty(0,T;\mathrm{BV}(\mathcal{D}))} + \|p^+\|_{L^\infty(0,T;\mathrm{BV}(\mathcal{D}))} \\
&\quad \le \mathcal{C}_{\mathrm{BV}}(T, \|p_0^\pm\|_{\mathrm{BV}(\mathcal{D})}, \|\mathbf{A}_\perp^0\|_{\mathbb{BV}(\mathbb{T}_L)}, \|\mathbf{A}_\perp^0{}'\|_{\mathbb{BV}(\mathbb{T}_L)}, \\
&\qquad \|\mathbf{A}_\perp^1\|_{\mathbb{BV}(\mathbb{T}_L)}, \|p^\pm\|_{L^\infty(Q)}, \|\partial_x \mathbf{A}_\perp\|_{\mathbb{L}^\infty(\Omega)}),
\end{aligned}$$

which ends the proof. □

3.3. *Global weak solutions*

We are now able to prove the global existence of weak solutions for the system (2.9)–(2.12). We have the following global existence theorem

Theorem 3.4. (Existence of global weak solutions) *Let us assume that* $p_0^\pm \in L^2 \cap L^\infty \cap \mathrm{BV}(\mathcal{D})$, *and* $\mathbf{A}_\perp^0$, $\mathbf{A}_\perp^0{}'$, $\mathbf{A}_\perp^1 \in \mathbb{L}^2 \cap \mathbb{L}^\infty \cap \mathbb{BV}(\mathbb{T}_L)$, *then the solution* $(p_\varepsilon^\pm, \phi_\varepsilon, \mathbf{A}_{\perp\varepsilon})$ *of the system* (3.1) *and* (2.10)–(2.12) *has a weak limit*

$$\begin{cases}
p^\pm \in \mathscr{C}(0,T; L^p(\mathcal{D})) \cap L^\infty(Q) \cap L^\infty(0,T; \mathrm{BV}(\mathcal{D})), \\
\phi \in \mathscr{C}(0,T; L^p(\mathbb{T}_L)) \cap W^{1,\infty}(\Omega) \cap L^\infty(0,T; W^{2,1}(\mathbb{T}_L)), \\
\mathbf{A}_\perp \in \mathscr{C}(0,T; \mathbb{L}^p(\mathbb{T}_L)) \cap \mathbb{W}^{1,\infty}(\Omega) \cap L^\infty(0,T; \mathbb{W}^{2,1}(\mathbb{T}_L)),
\end{cases} \tag{3.18}$$

for all $p \in [1, +\infty[$, *which satisfies the system* (2.9)–(2.12) *in* $\mathscr{D}'(Q)$ (*in sense of distribution*).

Proof. Since $p_\varepsilon^\pm \in L^\infty(0,T;\mathrm{BV}(\mathcal{D}))$, $\partial_x\phi_\varepsilon \in L^\infty(0,T;L^2(\mathbb{T}_L))$ and $\partial_x\mathbf{A}_{\perp\,\varepsilon} \in L^\infty(0,T;\mathbb{L}^2(\mathbb{T}_L))$, we have

$$\begin{cases}\partial_x p_\varepsilon^\pm \in L^\infty(0,T;\mathcal{M}_b(\mathcal{D})),\\ \partial_x\phi_\varepsilon \in L^\infty(0,T;\mathcal{M}_b(\mathbb{T}_L)) \quad \text{and} \quad \partial_x\mathbf{A}_{\perp\,\varepsilon} \in L^\infty(0,T;\mathbb{M}_b(\mathbb{T}_L)),\end{cases} \tag{3.19}$$

where $\mathcal{M}_b(K)$ is the space of bounded Radon measure on K, and $\mathbb{M}_b = \mathcal{M}_b \times \mathcal{M}_b$. Using (3.19) and since $|\partial_x\mathcal{H}_\varepsilon^\pm| \le |p_\varepsilon^\pm| + |\partial_x\mathbf{A}_{\perp\,\varepsilon}| + |\partial_x\phi_\varepsilon|$, we get $\partial_x\mathcal{H}_\varepsilon^\pm \in L^\infty(0,T;\mathcal{M}_b(\mathcal{D}))$. Since $\mathcal{M}_b(\mathcal{D}) \hookrightarrow W^{-\delta,p}(\mathcal{D})$, for $p \in (1,\frac{2}{2-\delta})$ and $\delta \in (0,2)$, the sequence $\{-\varepsilon\partial_x^2 p_\varepsilon^\pm + \partial_x\mathcal{H}_\varepsilon^\pm\}$ is bounded in $L^\infty(0,T;W^{-1-\delta,p}(\mathcal{D}))$ and using Eq. (3.1) we have $\{\partial_t p_\varepsilon^\pm\} \in L^\infty(0,T;W^{-1-\delta,p}(\mathcal{D}))$, which means that there exists a constant $\mathcal{C}_\flat$ independent of ε such that

$$\|p_\varepsilon^\pm(t) - p_\varepsilon^\pm(\tau)\|_{W^{-1-\delta,p}(\mathcal{D})} \le \mathcal{C}_\flat|t-\tau|, \quad \forall\, t,\,\tau > 0. \tag{3.20}$$

Since $\mathrm{BV}(\mathcal{D}) \hookrightarrow W^{1-\delta,p}(\mathcal{D})$, for $p \in (1,\frac{2}{2-\delta})$ and $\delta \in (0,2)$, we get

$$\|p_\varepsilon^\pm(t) - p_\varepsilon^\pm(\tau)\|_{W^{1-\delta,p}(\mathcal{D})} \le \|p_\varepsilon^\pm(t) - p_\varepsilon^\pm(\tau)\|_{\mathrm{BV}(\mathcal{D})} \le 2\mathcal{C}_{\mathrm{BV}}, \tag{3.21}$$

where $\mathcal{C}_{\mathrm{BV}}$ is independent of ε. Using (3.20) and (3.21) and the interpolation inequality $\|v\|_{W^{s,p}(\mathcal{D})} \le \mathcal{C}_\mathcal{I}\|v\|^{1-\sigma}_{W^{1-\delta,p}(\mathcal{D})}\|v\|^{\sigma}_{W^{-1-\delta,p}(\mathcal{D})}$, where $\sigma \in (0,1)$ is chosen small enough, and $s = 1-2\sigma-\delta > 0$, we then get $\|p_\varepsilon^\pm(t) - p_\varepsilon^\pm(\tau)\|_{W^{s,p}(\mathcal{D})} \le \mathcal{C}_\mathcal{I}(2\mathcal{C}_{\mathrm{BV}})^{1-\sigma}\mathcal{C}_\flat^\sigma|t-\tau|^\sigma$, which means that the sequence $\{p_\varepsilon^\pm\}$ is bounded in $\mathscr{C}^\sigma(0,T;W^{s,p}(\mathcal{D}))$ for all $T<\infty$ with $1 < p < \frac{2}{2-\delta} < \frac{2}{1+2\sigma}$. Therefore, noting that the Sobolev embedding $W^{s,p}(\mathcal{D}) \hookrightarrow L^p(\mathcal{D})$ is compact, the Ascoli's theorem implies that

$$\{p_\varepsilon^\pm\} \text{ is compact in } \mathscr{C}(0,T;L^p(\mathcal{D})), \quad \forall\, T < \infty. \tag{3.22}$$

Let $T<\infty$ be fixed, using (3.22), we can extract a subsequence $\{p_{\varepsilon_n}^\pm\}$ (with $\varepsilon_n \to 0$, as $n \to +\infty$), still noted $\{p_\varepsilon^\pm\}$ such that

$$\begin{cases} p_\varepsilon^\pm \to p^\pm & \text{in } \mathscr{C}(0,T;L^p(\mathcal{D})),\\ p_\varepsilon^\pm \to p^\pm & \text{a.e. on } Q.\end{cases} \tag{3.23}$$

Therefore, using a diagonal extraction procedure, we can get a subsequence still noted $\{p_\varepsilon^\pm\}$ such that (3.23) holds for all $T<\infty$. Now, since $\partial_t\mathbf{A}_{\perp\,\varepsilon}$ and $\partial_x\mathbf{A}_{\perp\,\varepsilon}$ belong to $L^\infty(0,T;\mathbb{L}^2(\mathbb{T}_L))$ there exist two constants $\mathcal{K}_\flat$ and $\mathcal{K}_c$ independent of ε such that for all $t,\tau > 0$,

$$\begin{cases}\|\mathbf{A}_{\perp\,\varepsilon}(t) - \mathbf{A}_{\perp\,\varepsilon}(\tau)\|_{\mathbb{L}^2(\mathbb{T}_L)} \le \mathcal{K}_\flat|t-\tau|,\\ \|\mathbf{A}_{\perp\,\varepsilon}(t) - \mathbf{A}_{\perp\,\varepsilon}(\tau)\|_{\mathbb{H}^1(\mathbb{T}_L)} \le \mathcal{K}_c.\end{cases} \tag{3.24}$$

From space interpolation result $[W^{\theta,q}(\mathbb{T}_L), W^{\nu,q}(\mathbb{T}_L)] = W^{\sigma\nu+(1-\sigma)\theta,q}(\mathbb{T}_L)$, for $\theta,\,\nu \in \mathbb{R}$, $\sigma \in (0,1)$, $q \in (1,\infty)$, and using (3.24) we obtain $\|\mathbf{A}_{\perp\,\varepsilon}(t) - \mathbf{A}_{\perp\,\varepsilon}(\tau)\|_{\mathbb{H}^{1-\sigma}(\mathbb{T}_L)} \le \mathcal{K}_\mathcal{I}\mathcal{K}_c^{1-\sigma}\mathcal{K}_\flat^\sigma|t-\tau|^\sigma$, which means that the sequence $\{\mathbf{A}_{\perp\,\varepsilon}\}$

is bounded in $\mathscr{C}^\sigma(0,T;\mathbb{H}^{1-\sigma}(\mathbb{T}_L))$. Since the Sobolev embedding $\mathbb{H}^{1-\sigma}(\mathbb{T}_L) \hookrightarrow \mathbb{L}^2(\mathbb{T}_L)$ is compact, the Ascoli's theorem implies that

$$\{\mathbf{A}_{\perp\,\varepsilon}\} \text{ is compact in } \mathscr{C}(0,T;\mathbb{L}^2(\mathbb{T}_L)), \quad \forall\, T<\infty. \tag{3.25}$$

Let $T<\infty$ be fixed, using (3.25), we can extract a subsequence $\{\mathbf{A}_{\perp\,\varepsilon_n}\}$ (with $\varepsilon_n \to 0$, as $n\to+\infty$), still noted $\{\mathbf{A}_{\perp\,\varepsilon}\}$ such that

$$\begin{cases} \mathbf{A}_{\perp\,\varepsilon} \to \mathbf{A}_\perp & \text{in } \mathscr{C}(0,T;\mathbb{L}^2(\mathbb{T}_L)),\\ \mathbf{A}_{\perp\,\varepsilon} \to \mathbf{A}_\perp & \text{a.e. on } \Omega. \end{cases} \tag{3.26}$$

Therefore, using a diagonal extraction procedure, we can get a subsequence still noted $\{\mathbf{A}_{\perp\,\varepsilon}\}$ such that (3.26) holds for all $T<\infty$. Assuming $T<\infty$ fixed, by linearity of the Poisson equation (2.11) and since $G\in L^\infty(\mathbb{T}_L\times\mathbb{T}_L)$ we get

$$\begin{aligned} \big|\|\phi\|_{\mathscr{C}(0,T;L^p(\mathbb{T}_L))} - \|\phi_\varepsilon\|_{\mathscr{C}(0,T;L^p(\mathbb{T}_L))}\big| &\le \|\phi-\phi_\varepsilon\|_{\mathscr{C}(0,T;L^p(\mathbb{T}_L))}\\ &\le C_L\|G\|_{L^\infty(\mathbb{T}_L\times\mathbb{T}_L)}\|p^\pm - p^\pm_\varepsilon\|_{\mathscr{C}(0,T;L^p(\mathbb{T}_L))}, \end{aligned}$$

which means that

$$\begin{cases} \phi_\varepsilon \to \phi & \text{in } \mathscr{C}(0,T;L^p(\mathbb{T}_L)),\\ \phi_\varepsilon \to \phi\phi & \text{a.e. on } \Omega. \end{cases} \tag{3.27}$$

Therefore, using a diagonal extraction procedure, we can get a subsequence still noted $\{\phi_\varepsilon\}$ such that (3.27) holds for all $T<\infty$. Moreover, we have

$$\|p^\pm_\varepsilon\|_{L^\infty(Q)}, \quad \|\phi_\varepsilon\|_{L^\infty(\Omega)}, \quad \|\mathbf{A}_{\perp\,\varepsilon}\|_{\mathbb{L}^\infty(\Omega)} < \infty. \tag{3.28}$$

The bounds (3.28) and (3.23) imply that (3.23) is true for $p\in[1,+\infty[$. Using (3.23) and (3.26)–(3.28), the Lebesgue dominated convergence theorem implies that $\mathcal{H}^\pm_\varepsilon \to \mathcal{H}^\pm$ in $L^p(Q)$ and thus $\partial_x\mathcal{H}^\pm_\varepsilon \to \partial_x\mathcal{H}^\pm$ in $\mathscr{D}'(Q)$ (in the distributional sense). In the same way properties (3.23), (3.26), (3.28) and the Lebesgue dominated convergence theorem implies that $\mathbf{A}_{\perp\,\varepsilon}\rho_{\gamma\,\varepsilon} \to \mathbf{A}_\perp\rho_\gamma$ in $L^1(\Omega)$, hence in $\mathscr{D}'(\Omega)$. Using (3.23) and (3.28) we have $p^\pm_\varepsilon \to p^\pm$ in $\mathscr{D}'(Q)$, hence $\partial_t p^\pm_\varepsilon \to \partial_t p^\pm$ in $\mathscr{D}'(Q)$ and $\varepsilon\partial_x^2 p^\pm_\varepsilon \to 0$ in $\mathscr{D}'(Q)$. Using (3.26) we have $\partial_t^2\mathbf{A}_{\perp\,\varepsilon} \to \partial_t^2\mathbf{A}_\perp$ in $\mathscr{D}'(\Omega)$ and $\partial_x^2\mathbf{A}_{\perp\,\varepsilon} \to \partial_x^2\mathbf{A}_\perp$ in $\mathscr{D}'(\Omega)$. Therefore the limit point $(p^\pm,\phi,\mathbf{A}_\perp)$ is a weak solution in $\mathscr{D}'(Q)$ of the system (2.9)–(2.12).

Now, let us prove that $p^\pm \in L^\infty(0,T;\mathrm{BV}(\mathcal{D}))$. Since $\{p^\pm_\varepsilon\}\in L^\infty(0,T;\mathrm{BV}(\mathcal{D}))$ and $p^\pm_\varepsilon \to p^\pm$ in $L^\infty(0,T;L^1_{\mathrm{loc}}(\mathcal{D}))$, we obtain $\|p^\pm\|_{L^\infty(0,T;\mathrm{BV}(\mathcal{D}))} \le \liminf_{\varepsilon\to0}\|p^\pm_\varepsilon\|_{L^\infty(0,T;\mathrm{BV}(\mathcal{D}))} \le \mathcal{C}_{\mathrm{BV}} < \infty$. Now, let us prove that $\phi\in L^\infty(0,T;W^{2,1}(\mathbb{T}_L))$ and $\mathbf{A}_\perp \in L^\infty(0,T;\mathbb{W}^{2,1}(\mathbb{T}_L))$. Since $p^\pm\in L^\infty(0,T;L^1(\mathcal{D}))$, using Poisson equation (2.12), we get $\phi\in L^\infty(0,T;W^{2,1}(\mathbb{T}_L))$. If $\Phi\in\mathscr{C}_0^\infty(\mathbb{T}_L)\times\mathscr{C}_0^\infty(\mathbb{T}_L)$

then, from d'Alembert integral representation formula, we get

$$\int_{\mathbb{T}_L} dx\, \partial_x \mathbf{A}_\perp \cdot \partial_x \Phi \leq (\|\mathbf{A}_\perp^{0\prime}\|_{\mathbb{BV}(\mathbb{T}_L)} + \|\mathbf{A}_\perp^1\|_{\mathbb{BV}(\mathbb{T}_L)})\|\Phi\|_{\mathbb{L}^\infty(\mathbb{T}_L)}$$
$$+ \frac{1}{2}\left| \int_0^t ds \int_0^1 da \int_{\mathbb{T}_L} dx\, (F(s, x+(t-s), a) - F(s, x-(t-s), a)) \cdot \partial_x \Phi \right|$$
$$\leq (\|\mathbf{A}_\perp^{0\prime}\|_{\mathbb{BV}(\mathbb{T}_L)} + \|\mathbf{A}_\perp^1\|_{\mathbb{BV}(\mathbb{T}_L)} + C_T \|F\|_{L^\infty(0,T;L^1(0,1;\mathbb{BV}(\mathbb{T}_L)))})\|\Phi\|_{\mathbb{L}^\infty(\mathbb{T}_L)}, \tag{3.29}$$

where

$$F(s, x \pm (t-s), a) = \int_{p^-(s,x\pm(t-s),a)}^{p^+(s,x\pm(t-s),a)} \frac{\mathbf{A}_\perp(s, x \pm (t-s))}{\sqrt{1+p^2+|\mathbf{A}_\perp(s, x\pm(t-s))|^2}} dp. \tag{3.30}$$

Now we have to check that $F \in L^\infty(0,T;L^1(0,1;\mathbb{BV}(\mathbb{T}_L)))$. Using Eq. (3.30) and obvious estimates we obtain

$$\tau_h F - F = F(s, x \pm (t-s) + h, a) - F(s, x \pm (t-s), a)$$
$$\leq 2(|\tau_h p^-| + |\tau_h p^+|)|\tau_h \mathbf{A}_\perp - \mathbf{A}_\perp| + |\tau_h p^- - p^-| + |\tau_h p^+ - p^+|,$$

and thus

$$\|\tau_h F - F\|_{L^\infty(0,T;\mathbb{L}^1(\mathcal{D}))} \leq h(2\left(\|p^-\|_{L^\infty(Q)} + \|p^+\|_{L^\infty(Q)}\right)\|\mathbf{A}_\perp\|_{L^\infty(0,T;\mathbb{BV}(\mathbb{T}_L))}$$
$$+ \|p^-\|_{L^\infty(0,T;\mathrm{BV}(\mathcal{D})} + \|p^+\|_{L^\infty(0,T;\mathrm{BV}(\mathcal{D}))}\big),$$

which proves (3.29). Finally using (3.29) we obtain

$$\left|\int_{\mathbb{T}_L} \partial_x^2 \mathbf{A}_\perp \cdot \Phi\, dx\right| = \left|\int_{\mathbb{T}_L} \partial_x \mathbf{A}_\perp \cdot \partial_x \Phi\, dx\right| \leq \mathcal{K}\left(\|p^\pm\|_{L^\infty(0,T;\mathrm{BV}(\mathcal{D})},\right.$$
$$\left.\|\mathbf{A}_\perp\|_{L^\infty(0,T;\mathbb{BV}(\mathbb{T}_L))}, \|\mathbf{A}_\perp^{0\prime}\|_{\mathbb{BV}(\mathbb{T}_L)}, \|\mathbf{A}_\perp^1\|_{\mathbb{BV}(\mathbb{T}_L)}\right)\|\Phi\|_{\mathbb{L}^\infty(\mathbb{T}_L)},$$

which proves that $\mathbf{A}_\perp \in L^\infty(0,T;\mathbb{W}^{2,1}(\mathbb{T}_L))$. □

Remark 3.1. Using integral formulation of Poisson (also Ampère equation) and waves equations, and regularity of the weak solution of Theorem 3.4, in the same spirit (by duality argument) of the end of the proof of Theorem 3.4, we can show that $\partial_{tx}^2 \mathbf{A}_\perp$, $\partial_t^2 \mathbf{A}_\perp \in L^\infty(0,T;\mathbb{L}^1(\mathbb{T}_L))$, hence $\partial_x \mathbf{A}_\perp \in \mathbb{W}^{1,1}(\Omega)$ and $\partial_{tx}^2 \phi \in L^\infty(0,T;L^1(\mathbb{T}_L))$, hence $\partial_x \phi \in W^{1,1}(\Omega)$.

3.4. *Order preserving solutions*

In this section we establish some order or monotonicity properties satisfied by the weak solution. We have the following theorem.

Theorem 3.5. *The global weak solutions of Theorem* 3.4 *are order preserving in the sense that for any* a, $b \in [0,1]$ *we have*

$$p_0^{\pm}(\cdot,a) \le p_0^{\pm}(\cdot,b) \Rightarrow p^{\pm}(\cdot,\cdot,a) \le p^{\pm}(\cdot,\cdot,b), \tag{3.31}$$

$$\operatorname{sign}(\partial_a p_0^{\pm}) = \operatorname{sign}(\partial_a p^{\pm}), \tag{3.32}$$

$$\operatorname{sign}(p_0^{+}(\cdot,a) - p_0^{-}(\cdot,b)) = \operatorname{sign}(p^{+}(\cdot,\cdot,a) - p^{-}(\cdot,\cdot,b)). \tag{3.33}$$

Proof. The proof is based on the Crandall–Tartar theorem concerning relations between nonexpansive and order preserving mappings.[24] Let us set $\widetilde{\omega}^{\pm} = p_1^{\pm} - p_2^{\pm} = p^{\pm}(t,x,a) - p^{\pm}(t,x,b)$, $\omega^{\pm} = \partial_a p^{\pm}$, and $\widetilde{\rho} = p^{+}(t,x,a) - p^{-}(t,x,b)$, then using Eq. (3.1) we obtain

$$\partial_t \widetilde{\omega}^{\pm} + \partial_x \left(\frac{p_1^{\pm} + p_2^{\pm}}{\gamma_1^{\pm} + \gamma_2^{\pm}}\, \widetilde{\omega}^{\pm} \right) = \varepsilon \partial_x^2 \widetilde{\omega}^{\pm}, \tag{3.34}$$

$$\partial_t \omega^{\pm} + \partial_x \left(\frac{p^{\pm}}{\gamma^{\pm}}\, \omega^{\pm} \right) = \varepsilon \partial_x^2 \omega^{\pm}, \tag{3.35}$$

$$\partial_t \widetilde{\rho} + \partial_x \left(\frac{p^{+}(a) + p^{-}(b)}{\gamma^{+}(a) + \gamma^{-}(b)}\, \widetilde{\rho} \right) = \varepsilon \partial_x^2 \widetilde{\rho}. \tag{3.36}$$

Let us treat the case of Eqs. (3.34) which lead to the property (3.31). The other two Eqs. (3.35) and (3.36), which lead respectively to the property (3.32) and (3.33), can be treated in the same way. Let $\zeta_h \in \mathscr{C}_0^{\infty}(\mathbb{R})$ be a convex regularization of the modulus function which converges uniformly to $|\cdot|$ as $h \to 0$ and satisfies $|\zeta_h'| \le 1$. If we multiply Eq. (3.34) by $\zeta_h'(\widetilde{\omega}^{\pm})$ and integrate in space variable x, using integration by parts we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{T}_L} \zeta_h(\widetilde{\omega}^{\pm}) &= -\int_{\mathbb{T}_L} dx\, \zeta_h'(\widetilde{\omega}^{\pm}) \partial_x \left(\frac{p_1^{\pm} + p_2^{\pm}}{\gamma_1^{\pm} + \gamma_2^{\pm}}\, \widetilde{\omega}^{\pm} \right) - \varepsilon \int_{\mathbb{T}_L} dx\, |\partial_x \widetilde{\omega}^{\pm}|^2 \zeta_h''(\widetilde{\omega}^{\pm}) \\ &\le -\int_{\mathbb{T}_L} dx\, \partial_x \left(\frac{p_1^{\pm} + p_2^{\pm}}{\gamma_1^{\pm} + \gamma_2^{\pm}} \right) \int_0^{\widetilde{\omega}^{\pm}} \zeta_h''(s) s ds \\ &\le \epsilon(h)(\|p_1^{\pm}\|_{L^\infty(0,T;\mathrm{BV}(\mathcal{D}))} + \|p_2^{\pm}\|_{L^\infty(0,T;\mathrm{BV}(\mathcal{D}))} + \|\partial_x \mathbf{A}_\perp\|_{\mathbb{L}^\infty(\Omega)}), \end{aligned} \tag{3.37}$$

where $\epsilon(h) \to 0$ as $h \to 0$. Passing to the limit in (3.37) as $h \to 0$ we obtain

$$\frac{d}{dt}\|\widetilde{\omega}^{\pm}\|_{L^1(\mathbb{T}_L)} \le 0, \tag{3.38}$$

which, after time integration, is equivalent to

$$\|p^{\pm}(t,\cdot,a) - p^{\pm}(t,\cdot,b)\|_{L^1(\mathbb{T}_L)} \le \|p_0^{\pm}(\cdot,a) - p_0^{\pm}(\cdot,b)\|_{L^1(\mathbb{T}_L)}. \tag{3.39}$$

If we now define the operators $\mathcal{T}^{\pm} : L_x^1(\mathbb{T}_L) \to L_x^1(\mathbb{T}_L)$ by $\mathcal{T}^{\pm} p_0^{\pm} = p^{\pm}$, obviously $\mathcal{T}^{\pm}$ are mapping in $L_x^1(\mathbb{T}_L)$ which conserve the integral, and are nonexpansive in $L^1_x(\mathbb{T}_L)$ thanks to property (3.38) or (3.39). Therefore using Proposition 1 of Ref. 24 the operators $\mathcal{T}^\pm$ are order preserving in the sense that properties (3.31) is satisfied. □

Remark 3.2. Let us note the fact the evolution operators $\mathcal{T}^\pm : L^1_x(\mathbb{T}_L) \to L^1_x(\mathbb{T}_L)$ are nonexpansive in $L^1_x(\mathbb{T}_L)$ does not imply a contraction principle in $L^1_{xa}(\mathcal{D})$ for the waterbag continuum $p^\pm$. The nonexpansiveness property involves the same solution $p^\pm$, with the same initial data (leading to the removal of electromagnetic field comparison terms in Eqs. (3.34)–(3.36)), and gives a supplementary information on the structure of the solution, namely the monotonicity of the waterbag continuum with respect to the a-variable. On the contrary the L^1-contraction principle should compare two different solutions in L^1_{xa}-norm, with two different initial conditions. While a L^1_{xa}-contraction principle would have been useful to show uniqueness, as it usually done for hyperbolic conservation laws, here we need to establish a stability property with respect to the initial condition to show uniqueness. In fact, in the case of "weakly" (i.e. by the means of a mean field, for instance the electromagnetic field) coupled system of first-order conservation laws, new source terms (including comparison terms between fields) arise and convert the L^1-contraction principle into a L^1-comparison principle leading to the L^1-stability property of the solutions with respect to their initial data (see Theorem 3.7).

3.5. *Uniqueness of solutions*

In this section we show uniqueness of the global weak solutions whose existence have been proved in Theorem 3.4. To prove uniqueness of the global weak solutions of the system (2.9)–(2.12), we should perform a Kruzkov's type analysis[18, 47, 48, 66] based on Kruzkov's entropy inequalities with particular entropies $\eta(\cdot) = |\cdot - \xi|$, the so-called Kruzkov's entropies. Nevertheless, for heterogeneous scalar conservation laws Kruzkov's analysis uses Taylor expansion ingredients and thus requires high regularity assumptions of heterogeneousness both in time and space, typically twice continuously differentiable flux functions in time and space. Roughly speaking, our flux functions $\mathcal{H}^\pm(\cdot,\cdot,p)$ are such that $\partial_x\mathcal{H}^\pm(\cdot,\cdot,p)$ are at most in $W^{1,1}_{tx}$ for all $p \in \mathbb{R}$. In order to show uniqueness with lower regularity assumption, we could try to use the concept of measure valued solution introduced by Diperna.[28, 62] Nevertheless the lack of regularity information of Young measures with respect to the space variable makes this method more difficult to use. Finally the kinetic formulation of conservation laws developed by many authors in Refs. 59, 61, 54, 55, 60, 25, 19, 20, 21, 36, 64 and 65 yields a simpler framework for showing uniqueness. Even if measure-valued solutions using Young measures tool are closely related to kinetic formulation,[59] this latter tool deals with simpler topological objects such that essentially bounded functions instead of Young measures. The fact that kinetic formulation is an efficient way to prove uniqueness result is now well known but not in the context of coupled systems like here. To the best of my knowledge there is no result concerning "self-consistently"-coupled hyperbolic conservation laws, i.e. when temporal and spatial inhomogeneities of the flux functions depend "weakly" of the main unknowns through a mean field (for instance the electromagnetic field $(\phi,\ \mathbf{A}_\perp)$) which itself satisfies a set of partial differential equations, involving the main unknowns for the definition of their source terms. While kinetic formulation usually allows to show uniqueness in a general L^1-regularity framework, here the case of coupled systems leads to the estimate of new comparison terms, including comparison terms between fields (terms (3.75) and (3.79)–(3.81) below), for which BV regularity of the main unknowns is required. Moreover, the kinetic formulation formalism allows to make the link between the weak solution of the relativistic waterbag continuum and the relativistic Vlasov–Maxwell equations with kinetic entropy defect measure. The following uniqueness proof is based on the ideas developed in Refs. 25, 60, 59 and 54.

3.5.1. *Entropy solutions and kinetic formulation*

Let us first recall the notion of entropy solution introduced by Kruzkov[47,48] for the relativistic waterbag continuum system (2.9)–(2.12), which allows to recover uniqueness of weak solutions. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a convex function which is twice continuously differentiable. Multiplying (3.1) by $\eta(p_\varepsilon^\pm)$, we obtain

$$\begin{aligned}\partial_t\eta(p_\varepsilon^\pm) + \partial_x q(t,x,p_\varepsilon^\pm) - (\partial_x q)(t,x,p_\varepsilon^\pm) + \eta'(p_\varepsilon^\pm)(\partial_x\mathcal{H})(t,x,p_\varepsilon^\pm)\\ = \varepsilon\partial_x^2\eta(p_\varepsilon^\pm) - \varepsilon\eta''(p_\varepsilon^\pm)|\partial_x p_\varepsilon^\pm|^2,\end{aligned} \tag{3.40}$$

where

$$q(t,x,p_\varepsilon^\pm) = \int_k^{p_\varepsilon^\pm} \eta'(p)\partial_p\mathcal{H}(t,x,p)dp,$$

$$(\partial_x q)(t,x,p_\varepsilon^\pm) = \int_k^{p_\varepsilon^\pm} \eta'(p)\partial_{px}^2\mathcal{H}(t,x,p)dp,$$

with the Hamiltonian $\mathcal{H}$, given by Eq. (2.13). Using (3.23) we have $\partial_t\eta(p_\varepsilon^\pm) \to \partial_t\eta(p^\pm)$ in $\mathscr{D}'(Q)$. From (3.23) and (3.26) we get $\partial_x q(t,x,p_\varepsilon^\pm) \to \partial_x q(t,x,p^\pm)$ in $\mathscr{D}'(Q)$. Using (3.23) and (3.26)–(3.28), the Lebesgue dominated convergence theorem implies that $\eta'(p_\varepsilon^\pm)(\partial_x\mathcal{H})(t,x,p_\varepsilon^\pm) \to \eta'(p^\pm)(\partial_x\mathcal{H})(t,x,p^\pm)$ and $(\partial_x q)(t,x,p_\varepsilon^\pm) \to (\partial_x q)(t,x,p^\pm)$ in $\mathscr{D}'(Q)$. Moreover, we have $\varepsilon\partial_x^2\eta(p_\varepsilon^\pm) \to 0$ in $\mathscr{D}'(Q)$. Finally using the convexity of the entropy η, we can pass to the limit in (3.40) as $\varepsilon \to 0$ to obtain the following entropy inequality in $\mathscr{D}'(Q)$

$$\partial_t\eta(p^\pm) + \partial_x q(t,x,p^\pm) - (\partial_x q)(t,x,p^\pm) + \eta'(p^\pm)(\partial_x\mathcal{H})(t,x,p^\pm) \le 0. \tag{3.41}$$

Definition 3.1. The triplet $(p^\pm,\phi,\mathbf{A}_\perp)$ satisfying regularity assumptions (3.18) is an entropy solution of the system (2.9)–(2.12) if it is a solution of (2.9)–(2.12) in $\mathscr{D}'(Q)$ (whose existence has been proved in Theorem 3.4) and if it satisfies the entropy inequality (3.41) in $\mathscr{D}'(Q)$ for all convex function $\eta \in \mathscr{C}^2(\mathbb{R})$.

Before going further, let us define the function $\chi:\mathbb{R}^2 \to \{-1,0,1\}$ as follows: $\chi(\xi,p) := 1$, if $0 < \xi < p$, $\chi(\xi,p) := -1$, if $p < \xi < 0$ and $\chi(\xi,p) := 0$ otherwise. The first theorem establishes the equivalence between the entropy solution of Definition 3.1 and the kinetic formulation of the relativistic waterbag continuum system (2.9)–(2.12).

Theorem 3.6. *Let the triplet* $(p^{\pm},\phi,\mathbf{A}_{\perp})$ *be a global weak solution of the system* (2.9)–(2.12), *given by Theorem* 3.4 *which satisfies regularity properties* (3.18). *Then it is an entropy solution if and only if there exists non-negative measures* $m^{\pm}(t,x,a,p)$ *such that* $m^{\pm}((0,T)\times\mathcal{D}\times\mathbb{R}) < +\infty$ *for all* $T > 0$, *and such that kinetic functions* $\chi(p,p^{\pm})$ *satisfy the following kinetic equations in* $\mathscr{D}'(\Sigma)$

$$\begin{cases} \partial_t\chi(p,p^{\pm}) + \nabla_{x,p}\cdot(\mathcal{F}(t,x,p)\chi(p,p^{\pm})) = \partial_p m^{\pm}(t,x,a,p) + \delta(p)\mathcal{S}(t,x,p), \\ \chi(p,p^{\pm}(t=0)) = \chi(p,p_0^{\pm}), \end{cases} \tag{3.42}$$

while the electromagnetic field $(\phi,\mathbf{A}_{\perp})$ *satisfies Eqs.* (2.10)–(2.12) *in* $\mathscr{D}'(\Omega)$, *where the source terms* (*densities of charge and current*) *are given by*

$$\rho_{\gamma}(t,x) = \int_0^1 da\int_{\mathbb{R}}(\chi(p,p^+) - \chi(p,p^-))\frac{dp}{\gamma(t,x,p)},$$

$$\rho(t,x) = \int_0^1 da\int_{\mathbb{R}}(\chi(p,p^+) - \chi(p,p^-))dp,$$

$$J_x(t,x) = \int_0^1 da\int_{\mathbb{R}}(\chi(p,p^+) - \chi(p,p^-))\partial_p\mathcal{H}dp.$$

In the above we have used the following notations $\mathcal{F} = (\mathcal{F}_x,\mathcal{F}_p)^T$, $\mathcal{F}_x = \partial_p\mathcal{H}$, $\mathcal{F}_p = -\partial_x\mathcal{H}(\nabla_{x,p}\cdot\mathcal{F} = 0)$, *and* $\mathcal{S}(t,x,p) = \mathcal{F}_p$.

Proof. Theorem 3.6 states the equivalence between the system constituted of Eqs. (2.9)–(2.12) and inequality (3.41) in $\mathscr{D}'$, whose existence of solution endowed with regularity properties (3.18) have been proved in Theorem 3.4, and the system constituted by Eqs. (3.42) and (2.10)–(2.12) in $\mathscr{D}'$. Since equations for the electromagnetic field remain identical in both formulation, i.e. especially Eqs. (2.10)–(2.12) are satisfied in $\mathscr{D}'(\Omega)$, it remains to show the equivalence in $\mathscr{D}'(Q)$ of Eqs. (2.9) and (3.41) on one hand and Eqs. (3.42) on the other hand. For this purpose we define the distributions $m^{\pm}$, which are solutions of (3.42), and we show that the distributions $m^{\pm}$ are non-negative and locally bounded if and only if $(p^{\pm},\phi,\mathbf{A}_{\perp})$ is an entropy solution characterized by Definition 3.1. Let us first define $m^{\pm}$. Since $p^{\pm}\in\mathscr{C}(0,T;L^1(\mathcal{D}))\cap L^{\infty}(Q)$, $\mathcal{F}\in L^{\infty}(\mathfrak{Q})$, and for a.e. $(t,x)\in\Omega$, $\mathcal{S}(t,x,\cdot)\in\mathscr{C}_b^{\infty}(\mathbb{R})$, we can define the distributions $m^{\pm}$ as

$$m^{\pm}(t,x,a,p) := \partial_t\int_0^p \chi(\xi,p^{\pm})dp + \partial_x\int_0^p \chi(\xi,p^{\pm})\mathcal{F}_x(t,x,\xi)d\xi + \mathcal{F}_p(t,x,p)\chi(p,p^{\pm}) - \int_0^p \delta(\xi)\mathcal{S}(t,x,\xi)d\xi. \tag{3.43}$$

By taking the distributional derivative of (3.43) with respect to p-variable, we obtain (3.42) in $\mathscr{D}'(\Sigma)$. If we multiply (3.42) by $\eta'(p)$, with $\eta' \in \mathscr{D}(\mathbb{R})$, integrate with respect to p, we obtain

$$\begin{aligned} &\partial_t \eta(p^\pm) + \partial_x q(t,x,p^\pm) - (\partial_x q)(t,x,p^\pm) + \eta'(p^\pm)(\partial_x \mathcal{H})(t,x,p^\pm) \\ &\quad = -\int_{\mathbb{R}} \eta''(p) m^\pm dp, \end{aligned} \tag{3.44}$$

where we have used the identity

$$\int_{\mathbb{R}} \psi'(\xi)\chi(\xi,p)d\xi = \psi(p) - \psi(0), \quad \forall\, p \in \mathbb{R}, \quad \forall\, \psi \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}), \tag{3.45}$$

and the fact that

$$\begin{aligned} &\int_{\mathbb{R}} \{\delta(p)\mathcal{S}(t,x,p)\eta'(p) + \eta''(p)\chi(p,p^\pm)\mathcal{F}_p\}dp \\ &\quad = -\eta'(p^\pm)(\partial_x \mathcal{H})(t,x,p^\pm) + (\partial_x q)(t,x,p^\pm). \end{aligned} \tag{3.46}$$

Now let us prove that $(p^\pm, \phi, \mathbf{A}_\perp)$ is an entropy solution if $m^\pm \geq 0$ are locally bounded measures. We assume that $m^\pm$ are non-negative measures on Σ such that $\int_\Sigma m^\pm dtdxdadp < \infty$, for all $T > 0$ and we set $\mathcal{B} = \max(\mathcal{B}^-, \mathcal{B}^+)$ with $\mathcal{B}^\pm := \|p^\pm\|_{L^\infty(Q)} < \infty$. In short, we want to extend (3.44) from $\eta' \in \mathscr{D}(\mathbb{R})$ to $\eta \in \mathscr{C}^2(\mathbb{R})$ and convex, so that convexity of η and non-negativity of $m^\pm$ lead to (3.41). To this aim, we first extend (3.44) for all $\eta \in \mathscr{C}^2(\mathbb{R})$ subquadratic, i.e. such that η'' is bounded. If $\eta \in \mathscr{C}^\infty(\mathbb{R})$ is subquadratic, we truncate η with a cutoff function $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$, such that $\varphi(p) = 1$, for $|p| \leq 1$, and $\varphi(p) = 0$, for $|p| \geq 2$. We set $\varphi_R(p) = \varphi(p/R)$, $\eta_R = \eta\varphi_R$. Therefore $\eta_R \in \mathscr{D}(\mathbb{R})$ and (3.44) is satisfied for η_R. Now for $R > \mathcal{B}$, we have $\eta_R(p^\pm) = \eta(p^\pm)$ and $q_R(t,x,p^\pm) = q(t,x,p^\pm)$ for a.e. $(t,x,a) \in Q$ so that (3.44) holds in $\mathscr{D}'(Q)$ when we replace the right of (3.44) by $-\int_{\mathbb{R}} \eta_R''(p)m^\pm dp$. It remains to notice that $\eta_R'' = \eta\varphi''(p/R)/R^2 + (2/R)\eta'\varphi'(p/R) + \eta''\varphi(p/R)$. Since $\varphi(p/R)$ monotically converges to one, and thanks to the bounds on η'', we can pass to the limit as $R \to \infty$ to obtain $\int_{\mathbb{R}} \eta''\varphi_R m^\pm dp \to \int_{\mathbb{R}} \eta'' m^\pm dp$. In a similar way the contribution of the first two terms vanishes in the limit $R \to \infty$. Equation (3.44) is then true for all $\eta \in \mathscr{C}^\infty(\mathbb{R})$ subquadratic. Similarly Eq. (3.44) can be generalized to all functions $\eta \in \mathscr{C}^2(\mathbb{R})$ convex and subquadratic by mollifying η and passing to the limit, thanks to the bounds on η, $m^\pm$ and $p^\pm$. Let us now release the subquadratic constraint by assuming $\eta \in \mathscr{C}^2(\mathbb{R})$ and convex. Since $\mathcal{B}^\pm < \infty$, we can change the values of $\eta(p)$ for large p to obtain a subquadratic function $\tilde{\eta}$ as follows. If we set $M = \sup_{|p|\leq \mathcal{B}} |\eta''(p)|$, we then define the $\mathscr{C}^2$-convex subquadratic function $\tilde{\eta}$ such that $\tilde{\eta}'' = \inf\{\eta'', M\}$ and $\tilde{\eta} = \eta$ for $|p| \leq \mathcal{B}$. We then have $\eta(p^\pm) = \tilde{\eta}(p^\pm)$, $\eta'(p^\pm) = \tilde{\eta}'(p^\pm)$ and $q(t,x,p^\pm) = \tilde{q}(t,x,p^\pm)$ a.e. in Q, so that (3.44) holds for $\tilde{\eta}$. The convexity of $\tilde{\eta}$ implies that $-\int_{\mathbb{R}} \tilde{\eta}''(p)m^\pm dp < 0$, hence entropy inequality is satisfied.

Conversely, let us prove that if now $(p^\pm, \phi, \mathbf{A}_\perp)$ is an entropy solution, then $m^\pm$ are non-negative locally bounded measures. If $\eta \in \mathscr{C}^\infty(\mathbb{R})$, is a convex function, such that $\eta' \in \mathscr{D}(\mathbb{R})$, then by comparing (3.41) and (3.44) we have $\int_{\mathbb{R}} \eta''(p) m^{\pm} dp \geq 0$. Nevertheless, there is no function η such that $\eta'' = \varphi$ where φ is a non-negative function in $\mathscr{D}_{+}(\mathbb{R})$. Thus once again, we can change the values of $\varphi(p)$ for large p to construct such a function. To achieve this construction, we have to study the behavior of $m^{\pm}$ for large p. In fact for $|p| > \mathcal{B}$, Eq. (3.43) becomes $\partial_t p^{\pm} + \partial_x \mathcal{H} = m^{\pm}$. If we compare this latter equation with Eq. (2.9) we obtain that $m^{\pm} = 0$ in $\mathscr{D}'((0,T) \times \mathcal{D} \times (\mathbb{R} \backslash [-\mathcal{B}, \mathcal{B}]))$, hence $m^{\pm}$ are locally bounded measures. If we now take $\varphi \in \mathscr{D}_{+}(\mathbb{R})$ and define $\eta \in \mathscr{C}^{\infty}$ such that $\eta'' = \varphi$, then η is a convex function. If we now construct $\tilde{\eta} \in \mathscr{D}(\mathbb{R})$ by multplying η with a cutoff function which is equal to one over the interval $[-\mathcal{B}, \mathcal{B}]$, then $\eta(p^{\pm}) = \tilde{\eta}(p^{\pm})$, $\eta'(p^{\pm}) = \tilde{\eta}'(p^{\pm})$ and $q(t,x,p^{\pm}) = \tilde{q}(t,x,p^{\pm})$ on Q. Moreover, Eq. (3.44) (respectively, (3.41)) holds for $\tilde{\eta}$ (respectively, η), and $\int_{\mathbb{R}} \varphi m^{\pm} dp = \int_{\mathbb{R}} \tilde{\eta}'' m^{\pm} dp$. Using these properties, the comparaison of (3.44) and (3.41) leads to

$$\int_{\mathbb{R}} \tilde{\eta}'' m^{\pm} dp = \int_{\mathbb{R}} \eta'' m^{\pm} dp = \int_{\mathbb{R}} \varphi m^{\pm} dp \geq 0, \quad \forall\, \varphi \in \mathscr{D}_{+}(\mathbb{R}),$$

which means that $m^{\pm}$ are non-negative measures on Σ. □

Remark 3.3. According to the equivalence Theorem 3.6, existence of entropy weak solutions (cf. Definition 3.1) for the system constituted of Eqs. (2.9)–(2.12) and inequality (3.41) in $\mathscr{D}'$, implies the existence of kinetic solutions for the system constituted of Eqs. (3.42) and (2.10)–(2.12) in $\mathscr{D}'$. The existence of entropy weak solutions with regularity properties (3.18) have been proved in Theorem 3.4. One could also prove directly existence of kinetic solutions for Eqs. (3.42) and (2.10)–(2.12), and use Theorem 3.6 to deduce existence of entropy weak solutions for Eqs. (2.9)–(2.12) and (3.41). To this aim, we can use the ideas developed in Refs. 60, 59, 54 and 25 which follow the Boltzmann approach to classical gas dynamics (hydrodynamic limit as the mean free path is vanishing). The procedure would start by studying the global strong solutions of the Vlasov–Maxwell equations constituted of field Eqs. (2.10)–(2.13) and kinetic transport equations

$$\partial_t f^{\pm} + \nabla_{x,p} \cdot (f^{\pm} \mathbb{J} \nabla_{x,p} \mathcal{H}) + \lambda f^{\pm} = g^{\pm} + h, \quad \text{with } \mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \lambda > 0, \tag{3.47}$$

and where the source term $g^{\pm} = g^{\pm}(t,x,a,p)$ is a given function and $h = -(\partial_x \mathcal{H})(t,x,0)$. The coupling between field equations (2.10)–(2.12) and kinetic transport equations (3.47) are obtained by the definition of the following charge and current densities $\rho_{\gamma} = \int_0^1 da \int_{\mathbb{R}} dp\, (f^{+} - f^{-})/\gamma$, $\rho = \int_0^1 da \int_{\mathbb{R}} dp\, (f^{+} - f^{-})$, and $J_x = \int_0^1 da \int_{\mathbb{R}} dp\, (f^{+} - f^{-}) \partial_p \mathcal{H}$. The superscripts $\pm$ and the parameter a can been seen as parameters describing different species of groups of particles in the plasma. Existence of the global strong solutions can be obtained in two steps.[37] The first one consists in reducing the problem to estimate the decrease of $f^{\pm}(t,x,a,p)$ when

$|p| \to \infty$ by using *a priori* bounds on distribution functions (L^p-norm, energy, ...), their first velocity moments (charge and current densities), fields (using integral representation formula for the fields) and their derivatives, the method of characteristics, the construction of a fixed-point application associated to an iterative scheme in time, classical arguments of compactness and Cauchy sequences. The second step consists in obtaining the decrease of $f(t,x,a,p)$ with respect to momentum p at infinity (or the control of the p-momentum supports of $f^\pm$, i.e. supports of $f^\pm$ should remain compact if initially they are) from *a priori* estimates and the method of characteristics or the method of propagation of moments and dispersion estimates.[53] Afterwards Eqs. (3.47) allow one to build new kinetic equations which are obtained by setting $g^\pm = \lambda\,\chi(p, p^\pm(t,x,a))$ with $p^\pm(t,x,a) = \int_{\mathbb{R}} f^\pm(t,x,a,p)dp$, and can be viewed as approximations of kinetic equations (3.42). Existence of global strong solutions for these new kinetic equations can be achieved by using a Banach fixed-point theorem. These new kinetic equations can be interpreted as BGK-type relaxation models towards the equilibrium $\chi(p,p^\pm)$ where the relaxation parameter λ plays the role of a collisional frequency. The main problem is then to prove that Eqs. (3.47) admit solution of the form $f^\pm(t,x,a,p) = \chi(p,p^\pm(t,x,a))$ when the source term takes the form $-\lambda f^\pm + g^\pm = -\lambda(f^\pm - \chi(p,p^\pm)) = \partial_p m_\lambda^\pm$ while passing to the limit as $\lambda \to \infty$. Therefore Eqs. (3.42) are obtained by passing to the limit in the BGK-type relaxation models (3.47) as the relaxation parameter λ tends to infinity (hydrodynamic limit). The passage to the limit relies on strong compactness argument in L^1 (Ascoli's theorem, BV bounds or time and spatial equi-continuity in L^1 and uniform spatial equi-integrability). This way of proving existence of kinetic solutions for Eqs. (3.42) and (2.10)–(2.12), yields rather lengthy and tedious calculations than those of the alternative which consists in proving existence of entropy weak solutions of Eqs. (2.9)–(2.12) and use the equivalence Theorem 3.6.

3.5.2. *Kinetic entropy defect measure properties*

The kinetic entropy defect measures satisfy the following properties.

Proposition 3.1. *The measures $m^\pm$ defined in Theorem* (3.6) *satisfy*

(i) (*Total mass of the measures*): $\forall\, T > 0$

$$\int_\Sigma m^\pm(t,x,a,p)dtdxdadp \le \frac{1}{2}\|p_0^\pm\|_{L^2(\mathcal{D})} - \int_\Sigma (\partial_x \mathcal{H})(t,x,p)\chi(p,p^\pm)dtdxdadp. \tag{3.48}$$

(ii) *We have*

$$\int_0^T dt \int_{\mathcal{D}} m^\pm(t,x,a,p)dxda \le \mu_T^\pm(p) \in L_0^\infty(\mathbb{R}), \tag{3.49}$$

where $L_0^\infty(\mathbb{R})$ *is the set of bounded functions which vanish at infinity and*

$$\mu_T^\pm(p) := \mathbb{1}_{p\geq 0}\|(p_0^\pm - p)_+\|_{L^1(\mathcal{D})} + \mathbb{1}_{p\leq 0}\|(p_0^\pm - p)_-\|_{L^1(\mathcal{D})} + (\|\partial_x\phi\|_{L^\infty(\Omega)} + \|\partial_x \mathbf{A}_\perp\|_{L^\infty(\Omega)})|\{(t,x,a)\in Q;\ s.t.\ p^\pm > |p|\}|. \tag{3.50}$$

(iii) *We have* $m^\pm(t,x,a,p) = 0$, *for* $p > \sup_{(t,x,a)\in Q} p^\pm$ *and* $p < \inf_{(t,x,a)\in Q} p^\pm$.

(iv) *If we consider a domain* $\mathcal{O}$ *of* $\mathbb{R}_+ \times \mathcal{D}$, *then* $m^\pm = 0$, *on* $\mathcal{O}$ *whenever* $p^\pm \in \mathscr{C}\cap W^{1,1}(\mathcal{O})$.

Proof. In fact, since $m^\pm = 0$ in $\mathscr{D}'((0,T)\times\mathcal{D}\times(\mathbb{R}\backslash[-\mathcal{B},\mathcal{B}]))$, inequality (3.44) is true for all convex function $\eta\in\mathscr{C}^2(\mathbb{R})$. If we now write Eq. (3.44) for $\eta\in\mathscr{C}^2(\mathbb{R})$ a non-negative convex subquadratic function such that $\eta(0)=0$, then $\eta(p^\pm(t,\cdot,\cdot))\in L^1(\mathcal{D})$ for all $t>0$. Integrating Eq. (3.44) on Q, using (3.46) and the periodicity of $\mathcal{S}$ with respect to the x-variable we obtain

$$\begin{aligned}\int_\Sigma \eta'' m^\pm dtdxdadp &\leq \int_{\mathcal{D}} \eta(p_0^\pm) + \int_\Sigma (\eta'(p)\delta(p) + \eta''(p)\chi(p,p^\pm))\mathcal{F}_p dtdxdadp\\ &\leq \int_{\mathcal{D}} \eta(p_0^\pm) - \int_\Sigma (\partial_x\mathcal{H})(t,x,p)\eta''(p)\chi(p,p^\pm)dtdxdadp.\end{aligned} \tag{3.51}$$

If we take $\eta(p) = p^2/2$ we obtain (3.48).

Using (3.48) with Kruzkov's entropy $\eta(p) = (p-p_0)_+$ (respectively, $\eta(p) = (p-p_0)_-$) with $p_0>0$ (respectively, $p_0<0$) yields $\eta'(p) = \mathrm{H}(p-p_0)$ (respectively, $\eta'(p) = -\mathrm{H}(p_0-p)$), $\eta''(p) = \delta(p-p_0)$ (respectively, $\eta''(p)=\delta(p-p_0)$) and

$$\begin{aligned}&\int_0^T dt\int_{\mathcal{D}} m^\pm(t,x,a,p_0)dtdxda\\ &\quad\leq \|(p_0^\pm - p_0)_+\|_{L^1(\mathcal{D})} + \int_\Sigma (\mathrm{H}(p-p_0)\delta(p) + \delta(p-p_0)\chi(p,p^\pm))\mathcal{F}_p dtdxdadp\\ &\quad\leq \|(p_0^\pm - p_0)_+\|_{L^1(\mathcal{D})}\\ &\qquad + (\|\partial_x\phi\|_{L^\infty(\Omega)} + \|\partial_x\mathbf{A}_\perp\|_{L^\infty(\Omega)})|\{(t,x,a)\in Q;\ \text{s.t. } |p^\pm|>|p_0|\}|.\end{aligned}$$

For negative values of p_0 we use the same argument with entropy $\eta(p) = (p-p_0)_-$. Therefore we obtain the upper bound (3.49) and (3.50). Let us note that $\mu_T^\pm(p)$ vanish at infinity, thanks to the Lebesgue dominated convergence theorem, hence $\mu_T^\pm \in L_0^\infty(\mathbb{R})$. Notice also that $\mu_T^\pm\in L^1(\mathbb{R})$ and using (3.50) we get $\|\mu_T^\pm\|_{L^1(\mathbb{R})} \leq 2(\|p_0^\pm\|_{L^\infty(\mathcal{D})}\|p_0^\pm\|_{L^1(\mathcal{D})} + TL\|p^\pm\|_{L^\infty(Q)})$. To prove (iii) we choose $\eta(p)\equiv 0$ for $p\leq \sup_{(t,x,a)\in Q}p^\pm$ and strictly convex for $p>\sup_{(t,x,a)\in Q}p^\pm$. Therefore using (3.51) we get $\int_\Sigma \eta'' m^\pm dtdxdadp \leq 0$ which means that $m^{\pm} = 0$, $\forall\, p > \sup_{(t,x,a)\in Q} p^{\pm}$. Similarly we choose $\eta(p) \equiv 0$ for $p \geq \inf_{(t,x,a)\in Q} p^{\pm}$ and strictly convex for $p < \inf_{(t,x,a)\in Q} p^{\pm}$. Therefore using (3.51) we get $\int_{\Sigma} \eta'' m^{\pm} dtdxdadp \leq 0$ which means that $m^{\pm} = 0$, $\forall\, p < \inf_{(t,x,a)\in Q} p^{\pm}$. Finally property (iv) results from the fact that (3.41) is an equality for $p^{\pm} \in \mathscr{C} \cap W^{1,1}(\mathcal{O})$. □

3.5.3. *Uniqueness*

The proof of uniqueness is based on the regularization by convolution of the kinetic equations (3.42) in order to use the chain rule. We then define four mollifiers $\zeta^{\alpha} \in \mathscr{D}(\mathbb{R})$, $\zeta^{\alpha} \geq 0$, with $\alpha \in \{t, x, a, p\}$, such that $\int_{\mathbb{R}} \zeta^{\alpha}(u)du = 1$, ζ^{α} is even, $\zeta^{\alpha\prime}$ is odd, and $\operatorname{supp} \zeta^{\alpha} \subset [-1, 1]$. If we note $z = (z_1, z_2) = (x, p)$, we define $\zeta_{\varepsilon}(t, x, p, a) = \zeta_{\varepsilon_t}(t)\zeta_{\varepsilon_x}(x)\zeta_{\varepsilon_p}(p)\zeta_{\varepsilon_a}(a) = \zeta_{\varepsilon_t}(t)\zeta_{\varepsilon_z}(z)\zeta_{\varepsilon_a}(a)$ with $\varepsilon_z = \varepsilon_{z_1}\varepsilon_{z_2}$ and where $\zeta_{\varepsilon_{\alpha}}(\alpha) = \varepsilon_{\alpha}^{-1}\zeta^{\alpha}(\alpha/\varepsilon_{\alpha})$. In the following we will use the notations $f^{\pm} = f^{\pm}(t, x, p, a) = f^{\pm}(t, z, a) = \chi^{\pm} = \chi(p, p^{\pm}(t, x, a))$, $f_{\varepsilon}^{\pm} = \chi(p, p^{\pm}) * \zeta_{\varepsilon}$, and $m_{\varepsilon}^{\pm} = m^{\pm} * \zeta_{\varepsilon}$.

Before proving the uniqueness we need to establish a technical lemma and proposition.

Lemma 3.1. *The kinetic functions $f_{\varepsilon}^{\pm}$ satisfy in $\mathscr{D}'(\Sigma)$*

$$\partial_t f_{\varepsilon}^{\pm} + \nabla_{x,p} \cdot (\mathcal{F}(t, x, p) f_{\varepsilon}^{\pm}) = \partial_p m_{\varepsilon}^{\pm}(t, x, a, p) + (\delta(p)\mathcal{S}(t, x, p)) * \zeta_{\varepsilon} + \mathcal{R}_{\varepsilon}^{\pm}, \tag{3.52}$$

where $\lim_{\varepsilon\to 0} \mathcal{R}_{\varepsilon}^{\pm} = 0$ *in* $L^1_{\mathrm{loc}}(\Sigma)$ *for all* $T > 0$. *Moreover, for all* $\psi \in \mathscr{D}(Q)$ *we have*

$$\left| \int_Q m_{\varepsilon}^{\pm}(t, x, a, p)\psi(t, x, a)dtdxda \right| \leq 2\|\psi\|_{L^{\infty}(Q)} \sup_{\{\xi;\, |p-\xi| \leq \varepsilon_p\}} \mu_{T+\varepsilon_t}^{\pm}(\xi). \tag{3.53}$$

Proof. Since we have taken the convolution product of the kinetic equations (3.42) by ζ_{ε} we get

$$\mathcal{R}_{\varepsilon}^{\pm} = \nabla_{x,p} \cdot (\mathcal{F}(t, x, p) f_{\varepsilon}^{\pm}(t, x, p, a)) - \nabla_{x,p} \cdot (\mathcal{F}(t, x, p) f^{\pm}(t, x, p, a)) * \zeta_{\varepsilon}. \tag{3.54}$$

If we multiply (3.54) by the test function $\psi \in \mathscr{D}(\Sigma)$, integrate the obtained results over Σ, use integration by parts with respect to z-variable and use the fact that ζ_{ε_t}, ζ_{ε_a} are even and ζ'_{ε_x}, ζ'_{ε_p} are odd, we obtain $\langle \mathcal{R}_{\varepsilon}^{\pm}, \psi \rangle = \langle \nabla_{x,p} \cdot (\mathcal{F}) f_{\varepsilon}^{\pm}, \psi \rangle + \langle \mathcal{L}_{\varepsilon}^{\pm}, \psi \rangle$, where

$$\mathcal{L}_{\varepsilon}^{\pm}(t, z, a) = \int_{\mathbb{R}^4} dt'dz'da' (\mathcal{F}(t, z) - \mathcal{F}(t', z')) f^{\pm}(t', z', a') \cdot \nabla_z \zeta_{\varepsilon}(t - t', z - z', a - a').$$

Since $\nabla_z \cdot \mathcal{F} = 0$, we get $\mathcal{R}_{\varepsilon}^{\pm} = \mathcal{L}_{\varepsilon}^{\pm}$. Let us now make the following decomposition $\mathcal{L}_{\varepsilon}^{\pm} = \mathcal{L}_{\varepsilon}^{\pm 1} + \mathcal{L}_{\varepsilon}^{\pm 2}$, where

$$\mathcal{L}_{\varepsilon}^{\pm 1} = \int_{\mathbb{R}^4} dt'dz'da' \; f^{\pm}(t', z', a')(\mathcal{F}(t, z) - \mathcal{F}(t', z)) \cdot \nabla_z \zeta_{\varepsilon}(t - t', z - z', a - a'), \tag{3.55}$$

$$\mathcal{L}_\varepsilon^{\pm 2} = \int_{\mathbb{R}^4} dt'dz'da' \ f^\pm(t',z',a')(\mathcal{F}(t',z) - \mathcal{F}(t',z')) \cdot \nabla_z \zeta_\varepsilon(t-t',z-z',a-a'). \tag{3.56}$$

Let us first estimate (3.55). We define the compact set $\mathcal{K}_{tz} = \mathcal{K}_t \times \mathcal{K}_z \subset \mathfrak{Q}$, the compact set $\mathcal{K}_a \subset [0,1]$ and $\mathcal{K} = \mathcal{K}_{tz} \times \mathcal{K}_a$. Therefore, using some change of variable and Taylor expansion with integral remainder, we have

$$\begin{aligned}\|\mathcal{L}_\varepsilon^{\pm 1}\|_{L^1(\mathcal{K})} &\leq \|f^\pm\|_{L^\infty(\Sigma)} \int_{\mathcal{K}_{tz}} dtdz \int_{\mathcal{K}_a} da \int_{\mathbb{R}^2} dz'da' \, \frac{1}{\varepsilon_z} \\ &\quad \times \left| \int_{\mathbb{R}} dt'(\mathcal{F}(t,z) - \mathcal{F}(t',z))\zeta_{\varepsilon_t}(t-t') \right| \zeta_{\varepsilon_a}(a-a')|(\nabla_z \zeta_{\varepsilon_z})(z-z')| \\ &\leq \|f^\pm\|_{L^\infty(\Sigma)} \|\nabla_z \zeta^z\|_{L^1(\mathbb{R})} \frac{|\mathcal{K}_a|}{\varepsilon_z} \int_0^1 d\tau \int_{\mathcal{K}_{tz}} dtdz \\ &\quad \cdot \int_{|u|\leq 1} du \varepsilon_t |u| \zeta^t(u) |\partial_t \mathcal{F}(t - \varepsilon_t \tau u, z)| \\ &\leq \frac{\varepsilon_t}{\varepsilon_z} \|\nabla_z \zeta^z\|_{L^1(\mathbb{R})} \|\partial_t \mathcal{F}\|_{\mathbb{L}^1(\mathcal{K}_{tz}^{\varepsilon_t})} \|u\zeta^t(u)\|_{L^1(\mathbb{R})},\end{aligned} \tag{3.57}$$

where the compact set $\mathcal{K}_{tz}^{\varepsilon_t} = \mathcal{K}_t^{\varepsilon_t} \times \mathcal{K}_z$ tends to $\mathcal{K}_{tz}$ as $\varepsilon \to 0$. Since $\mathcal{F} \in L^1(\mathbb{R}_p, \mathbb{W}^{1,1}(\Omega))$ (see Remark 3.1), if we choose $\varepsilon_z \to 0$, and $\varepsilon_t \to 0$ such that $\varepsilon_t/\varepsilon_z \to 0$, then $\lim_{\varepsilon \to 0} \mathcal{L}_\varepsilon^{\pm 1} = 0$ in $L^1_{\text{loc}}(\Sigma)$. Let us now deal with the term (3.56). Using Taylor expansion with integral remainder, the term (3.56) can be decomposed as follows $\mathcal{L}_\varepsilon^{\pm 2} = \mathcal{L}_\varepsilon^{\pm 21} + \mathcal{L}_\varepsilon^{\pm 22}$, where

$$\begin{aligned}\mathcal{L}_\varepsilon^{\pm 21} &= \sum_{i,j=1}^{2} \int_{\mathbb{R}^4} dt'dz'da' \int_0^1 d\tau (\partial_{z'_j} \mathcal{F}_i(t',z'+\tau(z-z')) - \partial_{z'_j} \mathcal{F}_i(t',z')) \\ &\quad \cdot (z-z')_j f^\pm(t',z',a') \nabla_{z_i} \zeta_\varepsilon(t-t',z-z',a-a'),\end{aligned} \tag{3.58}$$

$$\begin{aligned}\mathcal{L}_\varepsilon^{\pm 22} &= \sum_{i,j=1}^{2} \int_{\mathbb{R}^4} dt'dz'da' \partial_{z'_j} \mathcal{F}_i(t',z') \\ &\quad \cdot (z-z')_j f^\pm(t',z',a') \nabla_{z_i} \zeta_\varepsilon(t-t',z-z',a-a').\end{aligned} \tag{3.59}$$

Using some change of variable we obtain for the term (3.58),

$$\begin{aligned}\|\mathcal{L}_\varepsilon^{\pm,21}\|_{L^1(\mathcal{K})} &\leq \sum_{i,j=1}^{2} \int_0^1 d\tau \int_{\mathcal{K}_t} dt \int_{|u|<1} du \int_{\mathcal{K}_a} da \int_{\mathcal{K}_t^{\varepsilon_t}} dt' \int_{\mathcal{K}_z^{\varepsilon_z}} dz' \\ &\quad \cdot \int_{\mathcal{K}_a^{\varepsilon_a}} da' |f^{\pm\prime}| |u \cdot \nabla_u \zeta^z(u)| |\partial_{z'_j} \mathcal{F}_i(t',z'+\varepsilon_z \tau u) \\ &\quad - \partial_{z'_j} \mathcal{F}_i(t',z')| \zeta_{\varepsilon_t}(t-t') \zeta_{\varepsilon_a}(a-a') \\ &\leq \|u \cdot \nabla_u \zeta^z(u)\|_{L^1(\mathfrak{D})} \sum_{i,j=1}^{2} \omega_{\mathcal{K}_{tz}^{\varepsilon_{tz}}}(\partial_{z_j} \mathcal{F}_i, \varepsilon_z),\end{aligned} \tag{3.60}$$

where we define the modulus of continuity $\omega_K(g,h) := \sup_{|z|\le h}\|g(\cdot,\cdot+z) - g(\cdot,\cdot)\|_{L^1(K)}$ for any compact set $K \subset \mathfrak{Q}$ and function $g \in L^1(0,T;W^{1,1}_{\rm loc}(\mathfrak{D}))$. Notice that the compact set $\mathcal{K}^{\varepsilon_{tz}}_{tz} = \mathcal{K}^{\varepsilon_t}_t \times \mathcal{K}^{\varepsilon_z}_z$ tends to $\mathcal{K}_{tz}$ as $\varepsilon \to 0$. Since $\mathcal{F} \in L^1(\mathbb{R}_p, \mathbb{W}^{1,1}(\Omega))$ (see Remark 3.1), we have $\lim_{\varepsilon\to 0}\mathcal{L}^{\pm 21}_\varepsilon = 0$ in $L^1_{\rm loc}(\Sigma)$. Let us now deal with the term (3.59). We first show that $\mathcal{L}^{\pm 22}_\varepsilon$ is bounded in $L^1_{\rm loc}(\Sigma)$. Since $\mathcal{F} \in L^1(\mathbb{R}_p, \mathbb{W}^{1,1}(\Omega))$ (see Remark 3.1), we have

$$\|\mathcal{L}^{\pm,22}_\varepsilon\|_{L^1(\mathcal{K})} \le \sum_{i,j=1}^{2} \|u\cdot\nabla_u \zeta^z(u)\|_{L^1(\mathfrak{D})}\|f^\pm\|_{L^\infty(\Sigma)}\|\partial_{z_j}\mathcal{F}_i\|_{L^1(\mathcal{K}_\varepsilon)} < \infty.$$

In order to conclude we just need to observe that $\lim_{\varepsilon\to 0}\mathcal{L}^{\pm 22}_\varepsilon = -\nabla_z\cdot(\mathcal{F}_i(t,z))f^\pm(t,z,a) = 0$ in $L^1_{\rm loc}(\Sigma)$, when $f^\pm$ and $\mathcal{F}$ are smooth, by using the parity of function ζ^α, and integration by parts. Indeed the general case follows by density argument using the above bound. Finally we have proved that $\lim_{\varepsilon\to 0}\mathcal{R}_\varepsilon = 0$ in $L^1_{\rm loc}(\Sigma)$. Let us show the estimate (3.53). Using convolution properties we obtain

$$\begin{aligned}
&\left|\int_Q m^\pm_\varepsilon(t,x,a,p)\psi(t,x,a)dtdxda\right| \\
&\quad\le 2\|\psi \overset{txa}{*} \zeta_{\varepsilon_{txa}}\|_{L^\infty(\mathbb{R}^3)}\|\zeta^p\|_{L^1(\mathbb{R})}\sup_{|u|\le 1}\int_0^{T+\varepsilon_t}dt\int_0^L dx\int_0^1 da m^\pm(t,x,a,p+\varepsilon_p u) \\
&\quad\le 2\|\psi\|_{L^\infty(Q)}\sup_{\{\xi;\,|p-\xi|\le\varepsilon_p\}}\mu^\pm_{T+\varepsilon_t}(\xi)
\end{aligned}$$

which ends the proof of the lemma. □

A fundamental property of the kinetic entropy defect measures $m^\pm$ saying that, roughly speaking the measures $m^\pm$ vanish at $p = p^\pm$, is given in the following lemma

Proposition 3.2. *For all test function $\psi \in \mathscr{D}(\mathcal{D}\times\mathbb{R}_p)$ we have*

$$\int_0^T\int_{\mathbb{R}^3} m^\pm_\varepsilon(t,x,a,p)\nu^\pm_\varepsilon(t,x,a,p)\psi(x,a,p)dtdxdadp \to 0, \tag{3.61}$$

as $\varepsilon \to 0$, where $\nu^\pm_\varepsilon = \nu^\pm * \zeta_\varepsilon$, with $\nu^\pm = \delta(p - p^\pm(t,x,a))$.

Proof. The proof relies on the comparison between $|f^\pm|^2 = |\chi^\pm|^2$ and $|f^\pm| = \mathrm{sign}(p)f^\pm = |\chi^\pm| = \mathrm{sign}(p)\chi^\pm$. We define $\mathrm{sign}_\varepsilon(p) = \mathrm{sign}(p) * \zeta_\varepsilon$ and $g^\pm_\varepsilon = \mathrm{sign}_\varepsilon f^\pm_\varepsilon$. Multiplying (3.52) by $\mathrm{sign}_\varepsilon$ gives

$$\partial_t g^\pm_\varepsilon + \nabla_z\cdot(\mathcal{F}g^\pm_\varepsilon) = \mathrm{sign}_\varepsilon\partial_p m^\pm_\varepsilon + \mathrm{sign}_\varepsilon\mathcal{R}^\pm_\varepsilon + [(\delta(p)\mathcal{S}(t,x,p)) * \zeta_\varepsilon]\mathrm{sign}_\varepsilon + \mathrm{sign}'_\varepsilon\mathcal{F}_p f^\pm_\varepsilon,$$

while the multiplication of (3.52) by $2f^\pm_\varepsilon$ leads to

$$\partial_t {f^\pm_\varepsilon}^2 + \nabla_z\cdot(\mathcal{F}{f^\pm_\varepsilon}^2) = 2f^\pm_\varepsilon\partial_p m^\pm_\varepsilon + 2f^\pm_\varepsilon\mathcal{R}^\pm_\varepsilon + 2[(\delta(p)\mathcal{S}(t,x,p)) * \zeta_\varepsilon]f^\pm_\varepsilon.$$

For all functions $\psi \in \mathscr{D}(\mathcal{D} \times \mathbb{R}_p)$, and for all $T > 0$, we have

$$\int_{\mathbb{R}^3} (f_\varepsilon^{\pm^2}(T) - g_\varepsilon^\pm(T))\psi dzda \tag{3.62}$$

$$= \int_{\mathbb{R}^3} (f_\varepsilon^{\pm^2}(0) - g_\varepsilon^\pm(0))\psi dzda \tag{3.63}$$

$$+ \int_0^T \int_{\mathbb{R}^3} (f_\varepsilon^{\pm^2} - g_\varepsilon^\pm)\mathcal{F} \cdot \nabla_z \psi dtdzda \tag{3.64}$$

$$+ \int_0^T \int_{\mathbb{R}^3} m_\varepsilon^\pm (\mathrm{sign}'_\varepsilon(p) - 2\partial_p f_\varepsilon^\pm)\psi dtdzda \tag{3.65}$$

$$+ \int_0^T \int_{\mathbb{R}^3} m_\varepsilon^\pm (\mathrm{sign}_\varepsilon(p) - 2 f_\varepsilon^\pm)\partial_p \psi dtdzda \tag{3.66}$$

$$+ \int_0^T \int_{\mathbb{R}^3} \mathcal{R}_\varepsilon^\pm (2 f_\varepsilon^\pm - \mathrm{sign}_\varepsilon(p))\psi dtdzda \tag{3.67}$$

$$+ \int_0^T \int_{\mathbb{R}^3} (2 f_\varepsilon^\pm (\delta(p)\mathcal{S}(t,x,p)) * \zeta_\varepsilon - \mathrm{sign}'_\varepsilon(p)\mathcal{F}_p f_\varepsilon^\pm)\psi dtdzda \tag{3.68}$$

$$- \int_0^T \int_{\mathbb{R}^3} \mathrm{sign}_\varepsilon(p)[(\delta(p)\mathcal{S}(t,x,p)) * \zeta_\varepsilon]\psi dtdzda. \tag{3.69}$$

Since $\mathrm{sign}'_\varepsilon(p) = 2\delta(p) * \zeta_\varepsilon$ and $\partial_p f_\varepsilon^\pm = (\delta(p) - \nu^\pm) * \zeta_\varepsilon$, we get $\mathrm{sign}'_\varepsilon(p) - 2\partial_p f_\varepsilon^\pm = 2\nu_\varepsilon^\pm$ for the term (3.65). We now choose $\psi = \psi_R(x,a,p) = \lambda(x)\beta_R(p)\theta(a)$, with $\lambda \in \mathscr{D}(\mathbb{T}_L)$, $\theta \in \mathscr{D}([0,1])$, $\beta_R \in \mathscr{D}(\mathbb{R})$; $\beta_R = 1$, if $|p| < R$, and $\beta_R = 0$, if $|p| > R+1$, $0 \le |\beta'_R| \le C$. We set $\varphi(x,a) = \lambda\theta$. Using (3.53), the term (3.66) can be estimated as

$$\int_0^T \int_{\mathbb{R}^3} m_\varepsilon^\pm (\mathrm{sign}_\varepsilon(p) - 2 f_\varepsilon^\pm)\varphi\beta'_R dtdzda \le 3C\|\varphi\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}} h_R^{\pm T}(p)dp,$$

where $h_R^{\pm T}(p) = (\mathbb{1}_{[R,R+1]}(p) + \mathbb{1}_{[-R-1,-R]}(p)) \sup_{\{\xi;\,|p-\xi|\le 1\}} \mu_{T+1}^\pm(\xi)$. Since $h_R^{\pm T}(p) \xrightarrow{R\to\infty} 0$ for a.e. $p \in \mathbb{R}$ (recall that $\mu_{T+1}^\pm \in L_0^\infty$) and $|h_R^{\pm T}| \le \sup_{\{\xi;\,|p-\xi|\le 1\}} \mu_{T+1}^\pm(\xi) \in L^1(\mathbb{R}_p)$, the Lebesgue dominated convergence theorem implies that $\lim_{R\to\infty} \int_{\mathbb{R}} h_R^{\pm T}(p)dp = 0$. Now, since $p^\pm$, $\partial_x\phi$, and $\partial_x \mathbf{A}_\perp \in L^\infty(Q)$ we have

$$\int_\Sigma |\chi(p,p^\pm)||\mathcal{F}_p(t,x,p)|dtdxdadp < \infty. \tag{3.70}$$

Using property (3.70) and from the fact that

$$\begin{cases} \lim\limits_{\varepsilon\to 0} g_\varepsilon^\pm = \mathrm{sign}(p) f^\pm = |\chi^\pm| \quad \text{in } L^1_{\mathrm{loc}}(\Sigma), \\ \lim\limits_{\varepsilon\to 0} f_\varepsilon^{\pm^2} = f^{\pm^2} = |\chi^\pm| \quad \text{in } L^1_{\mathrm{loc}}(\Sigma), \end{cases} \tag{3.71}$$

the terms (3.62)–(3.64) vanish as $\varepsilon \to 0$, thanks to the Lebesgue dominated convergence theorem. Using the properties

$$\begin{cases} \delta(p)\mathcal{S}(t,x,p) * \varphi_\varepsilon \xrightarrow{\varepsilon\to 0} \mathcal{S}(t,x,0) \quad \text{in } L^1_{\text{loc}}(\Sigma), \\ \mathcal{R}_\varepsilon \xrightarrow{\varepsilon\to 0} 0 \quad \text{in } L^1_{\text{loc}}(\Sigma) \text{ (Lemma 3.1)}, \\ \text{sign}_\varepsilon(p)[(\delta(p)\mathcal{S}(t,x,p)) * \zeta_\varepsilon] \xrightarrow{\varepsilon\to 0} 0 \quad \text{in } L^1_{\text{loc}}(\Sigma), \end{cases}$$

the terms (3.67)–(3.69) vanish as $\varepsilon \to 0$. Finally, using Lebesgue dominated convergence theorem we can pass to the limit as $\varepsilon \to 0$ in (3.62)–(3.69) to obtain

$$0 \le \lim_{\varepsilon\to 0} 2\int_0^T \int_{\mathbb{R}^3} m_\varepsilon^\pm \nu_\varepsilon^\pm(t,x,a,p)\psi_R dtdxdadp \le 3C\|\varphi\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}} h_R^{\pm\,T}(p)dp.$$

The above and Lebesgue dominated convergence theorem allows one to pass to the limit as $R \to \infty$ to get the desired result. □

We are now able to prove the uniqueness of the global weak solutions of Theorem 3.4, thanks to the L^1-stability result stated in the following Theorem 3.7. Let us note that the "self-consistently"-coupled waterbag continuum leads to the estimate of new terms (more precisely terms (3.75), and (3.79)–(3.81) below) involving comparison between electromagnetic fields generated by each solution.

Theorem 3.7. (Uniqueness) *The global weak solutions of Theorem* (3.4) *are unique. Moreover, if* $(p_i^\pm, \phi_i, \mathbf{A}_{\perp,i})$ *with* $i \in \{1,2\}$, *are two solutions then there exists a constant* $\mathcal{C}^\dagger$ *such that*

$$\begin{aligned} &\|p_1^-(T) - p_2^-(T)\|_{L^1(\mathcal{D})} + \|p_1^+(T) - p_2^+(T)\|_{L^1(\mathcal{D})} \\ &\quad \le \mathcal{C}^\dagger\{\|p_{0\,1}^- - p_{0\,2}^-\|_{L^1(\mathcal{D})} + \|p_{0\,1}^+ - p_{0\,2}^+\|_{L^1(\mathcal{D})} + \|\mathbf{A}^{0\,\prime}_{\perp,\,1} - \mathbf{A}^{0\,\prime}_{\perp,\,2}\|_{\mathbb{L}^1(\mathbb{T}_L)} \\ &\qquad + \|\mathbf{A}^{0}_{\perp,\,1} - \mathbf{A}^{0}_{\perp,\,2}\|_{\mathbb{L}^1\cap\mathbb{L}^\infty(\mathbb{T}_L)} + \|\mathbf{A}^{1}_{\perp,\,1} - \mathbf{A}^{1}_{\perp,\,2}\|_{\mathbb{L}^1\cap\mathbb{L}^\infty(\mathbb{T}_L)}\}, \end{aligned}$$

with the notation $\|\cdot\|_{\mathbb{L}^1\cap\mathbb{L}^\infty} = \|\cdot\|_{\mathbb{L}^1} + \|\cdot\|_{\mathbb{L}^\infty}$, *and where, for* $i = 1, 2$,

$$\mathcal{C}^\dagger := \mathcal{C}^\dagger(T, L, \|\mathbf{A}_{\perp,\,i}\|_{\mathbb{L}^\infty(\Omega)}, \|\partial_x\mathbf{A}_{\perp,\,i}\|_{\mathbb{L}^\infty(\Omega)}, \|p_i^\pm\|_{L^\infty(Q)}, \|p_i^\pm\|_{L^\infty(0,T;\text{BV}(\mathcal{D}))}).$$

Moreover, we have

$$\begin{aligned} &\|\phi_1(T) - \phi_2(T)\|_{L^1(\mathbb{T}_L)} \\ &\quad \le \|G\|_{L^1(\mathbb{T}_L)\times L^\infty(\mathbb{T}_L)}\{\|p_1^-(T) - p_2^-(T)\|_{L^1(\mathcal{D})} + \|p_1^+(T) - p_2^+(T)\|_{L^1(\mathcal{D})}\}, \end{aligned}$$

and

$$\begin{aligned} &\|\mathbf{A}_{\perp,\,1}(T) - \mathbf{A}_{\perp,\,2}(T)\|_{\mathbb{L}^1(\mathbb{T}_L)} \\ &\quad \le \mathcal{C}^\ddagger(T, L, \|p_1^\pm\|_{L^\infty(Q)}, \|p_2^\pm\|_{L^\infty(Q)})\{\|\mathbf{A}^0_{\perp,\,1} - \mathbf{A}^0_{\perp,\,2}\|_{\mathbb{L}^1(\mathbb{T}_L)} \\ &\qquad + \|\mathbf{A}^1_{\perp,\,1} - \mathbf{A}^1_{\perp,\,2}\|_{\mathbb{L}^1(\mathbb{T}_L)} + \|p_1^- - p_2^-\|_{L^\infty(0,T;L^1(\mathcal{D}))} + \|p_1^+ - p_2^+\|_{L^\infty(0,T;L^1(\mathcal{D}))}\}. \end{aligned}$$

Proof. Let us set $f^{\pm}_{\varepsilon,i} = \chi^{\pm}_{\varepsilon,i} = \chi(p, p^{\pm}_i) * \zeta_\varepsilon$ and $m^{\pm}_{\varepsilon,i} = m^{\pm}_i * \zeta_\varepsilon$, for $i = 1, 2$. Therefore we get

$$\partial_t (f^{\pm}_{\varepsilon,1} - f^{\pm}_{\varepsilon,2})^2 + \nabla_z \cdot (\mathcal{F}_1 (f^{\pm}_{\varepsilon,1} - f^{\pm}_{\varepsilon,2})^2) - \nabla_z \cdot ((\mathcal{F}_1 - \mathcal{F}_2) f^{\pm^2}_{\varepsilon,2})$$
$$+ 2 f^{\pm}_{\varepsilon,1} \nabla_z \cdot ((\mathcal{F}_1 - \mathcal{F}_2) f^{\pm}_{\varepsilon,2}) = 2(f^{\pm}_{\varepsilon,1} - f^{\pm}_{\varepsilon,2})\{\partial_p (m^{\pm}_{\varepsilon,1} - m^{\pm}_{\varepsilon,2})$$
$$+ (\mathcal{R}_{\varepsilon,1} - \mathcal{R}_{\varepsilon,2}) + (\delta(p)\mathcal{S}_1(t, x, p)) * \zeta_\varepsilon - (\delta(p)\mathcal{S}_2(t, x, p)) * \zeta_\varepsilon\}. \quad (3.72)$$

Let us construct $\psi = \psi_R \in \mathscr{D}(\mathbb{R}^3)$ as follows: $\psi_R = \lambda_R(x)\theta(a)\beta_R(p)$ where $\theta \in \mathscr{D}([0, 1])$ is such that $0 \le \theta \le 1$. The function β_R is chosen as in the proof of Proposition 3.2. We choose $\lambda \in \mathscr{D}_+(\mathbb{R})$ such that $\lambda(x) = 1$ for $|x| \le 1$ and we set $\lambda_R(x) = \lambda(x/R)$. Multiplying Eq. (3.72) by ψ_R, we obtain after integration

$$\int_{\mathcal{D}\times\mathbb{R}} (f^{\pm}_{\varepsilon,1}(T) - f^{\pm}_{\varepsilon,2}(T))^2 \psi_R dx da dp \quad (3.73)$$

$$= \int_{\mathcal{D}\times\mathbb{R}} (f^{\pm}_{\varepsilon,1}(0) - f^{\pm}_{\varepsilon,2}(0))^2 \psi_R dx da dp \quad (3.74)$$

$$+ \int_0^T \int_{\mathcal{D}\times\mathbb{R}} [\mathcal{F}_1 (f^{\pm}_{\varepsilon,1} - f^{\pm}_{\varepsilon,2})^2 - (\mathcal{F}_1 - \mathcal{F}_2) f^{\pm^2}_{\varepsilon,2}] \cdot \nabla_z \psi_R dt dx da dp \quad (3.75)$$

$$- 2 \int_0^T \int_{\mathcal{D}\times\mathbb{R}} (m^{\pm}_{\varepsilon,1} - m^{\pm}_{\varepsilon,2})(f^{\pm}_{\varepsilon,1} - f^{\pm}_{\varepsilon,2}) \partial_p \psi_R dt dx da dp \quad (3.76)$$

$$- 2 \int_0^T \int_{\mathcal{D}\times\mathbb{R}} (m^{\pm}_{\varepsilon,1} - m^{\pm}_{\varepsilon,2}) \partial_p (f^{\pm}_{\varepsilon,1} - f^{\pm}_{\varepsilon,2}) \psi_R dt dx da dp \quad (3.77)$$

$$+ 2 \int_0^T \int_{\mathcal{D}\times\mathbb{R}} (\mathcal{R}^{\pm}_{\varepsilon,1} - \mathcal{R}^{\pm}_{\varepsilon,2})(f^{\pm}_{\varepsilon,1} - f^{\pm}_{\varepsilon,2}) \psi_R dt dx da dp \quad (3.78)$$

$$+ 2 \int_0^T \int_{\mathcal{D}\times\mathbb{R}} [(\delta(p)\mathcal{S}_1(t, x, p)) * \zeta_\varepsilon - (\delta(p)\mathcal{S}_2(t, x, p)) * \zeta_\varepsilon]$$
$$\cdot (f^{\pm}_{\varepsilon,1} - f^{\pm}_{\varepsilon,2}) \psi_R dt dx da dp \quad (3.79)$$

$$- 2 \int_0^T \int_{\mathcal{D}\times\mathbb{R}} f^{\pm}_{\varepsilon,1} \nabla_z \cdot ((\mathcal{F}_1 - \mathcal{F}_2) f^{\pm}_{\varepsilon,2}) \psi_R dt dx da dp. \quad (3.80)$$

As in the proof of Proposition 3.2, using the Lebesgue dominated convergence theorem and the property (3.71) we can pass to the limit first as $\varepsilon \to 0$ and secondly as $R \to \infty$ in the terms (3.73)–(3.74) to obtain

$$\int_{\mathcal{D}\times\mathbb{R}} |f^{\pm}_1(T) - f^{\pm}_2(T)|^2 dx da dp = \|p^{\pm}_1(T) - p^{\pm}_2(T)\|_{L^1(\mathcal{D})},$$
$$\int_{\mathcal{D}\times\mathbb{R}} |f^{\pm}_{0\,1} - f^{\pm}_{0\,2}|^2 dx da dp = \|p^{\pm}_{0\,1} - p^{\pm}_{0\,2}\|_{L^1(\mathcal{D})}.$$

Since $\mathcal{F}_i \in \mathbb{L}^\infty(\mathfrak{Q})$ and $p^{\pm}_{\varepsilon,i} \in L^\infty(Q)$ for $i = 1, 2$, the Lebesgue dominated convergence theorem allows to pass to limit in the term (3.75), first as $\varepsilon \to 0$, secondly as $R \to \infty$, so that the term (3.75) vanishes. The term (3.76) can be estimated as follows:

$$-2\int_0^T\int_{\mathcal{D}\times\mathbb{R}}(m^\pm_{\varepsilon,1}-m^\pm_{\varepsilon,2})(f^\pm_{\varepsilon,1}-f^\pm_{\varepsilon,2})\partial_p\psi_R dtdxdadp$$
$$\leq C\int_{\mathbb{R}}\big(h^{\pm\,T}_{R,1}(p)+h^{\pm\,T}_{R,2}(p)\big)dp,$$

where $h^{\pm\,T}_{R,i}(p)$, for $i=1,2$, have same form as $h^{\pm\,T}_R$. Since $h^{\pm\,T}_{R,i}(p)\xrightarrow{R\to\infty}0$ for a.e. $p\in\mathbb{R}$ and $|h^{\pm\,T}_{R,i}|\leq\sup_{\{\xi;\,|p-\xi|\leq1\}}\mu^{\pm\,i}_{T+1}(\xi)\in L^1(\mathbb{R}_p)$, the Lebesgue dominated convergence theorem implies that the term (3.76) vanishes as $R\to\infty$. Using Lemma 3.1 and Lebesgue dominated convergence theorem the term (3.78) vanishes as $\varepsilon\to0$, $\forall\,R>0$. Using property (3.70) and Lebesgue dominated convergence theorem we can pass to limit as $\varepsilon\to0$, $\forall\,R>0$ in the term (3.79) so that it vanishes. Now since $\partial_p f^\pm_{\varepsilon,i}=\delta(p)*\zeta_\varepsilon-\delta(p-p^\pm_{\varepsilon,i})*\zeta_\varepsilon$, for $i=1,2$, using positivity of the measures $m^\pm_{\varepsilon,i}$ we get

$$-2\int_0^T\int_{\mathcal{D}\times\mathbb{R}}(m^\pm_{\varepsilon,1}-m^\pm_{\varepsilon,2})\partial_p(f^\pm_{\varepsilon,1}-f^\pm_{\varepsilon,2})\psi_R dtdxdadp$$
$$\leq 2\int_0^T\int_{\mathcal{D}\times\mathbb{R}}\{m^\pm_{\varepsilon,1}\delta(p-p^\pm_{\varepsilon,1})*\zeta_\varepsilon+m^\pm_{\varepsilon,2}\delta(p-p^\pm_{\varepsilon,2})*\zeta_\varepsilon\}\psi_R dtdxdadp,$$

which vanish as $\varepsilon\to0$, $\forall\,R>0$, thanks to Proposition 3.2. Using integration by parts, the term (3.80) can be decomposed as

$$-2\int_0^T\int_{\mathcal{D}\times\mathbb{R}}f^\pm_{\varepsilon,1}\nabla\cdot((\mathcal{F}_1-\mathcal{F}_2)f^\pm_{\varepsilon,2})\psi_R dtdxdadp$$
$$=2\int_0^T\int_{\mathcal{D}\times\mathbb{R}}\{f^\pm_{\varepsilon,1}f^\pm_{\varepsilon,2}(\mathcal{F}_1-\mathcal{F}_2)\cdot\nabla_z\psi_R+f^\pm_{\varepsilon,2}(\mathcal{F}_1-\mathcal{F}_2)\cdot\nabla_z f^\pm_{\varepsilon,1}\psi_R\}dtdxdadp. \tag{3.81}$$

Since $\mathcal{F}_i\in\mathbb{L}^\infty(\Omega)$ and $p^\pm_{\varepsilon,i}\in L^\infty(Q)$ for $i=1,2$, Lebesgue dominated convergence theorem allows one to pass to the limit, first as $\varepsilon\to0$, secondly as $R\to\infty$ in the first term of the right-hand side of (3.81) so that this latter term vanishes. For the second term of the right-hand side of (3.81) we get

$$2\int_0^T\int_{\mathcal{D}\times\mathbb{R}}f^\pm_{\varepsilon,2}(\mathcal{F}_1-\mathcal{F}_2)\cdot\nabla_z f^\pm_{\varepsilon,1}\psi_R dtdxdadp$$
$$\leq 2\int_0^T dt\int_{\mathcal{D}\times\mathbb{R}}dxdadp\int_{\mathbb{R}^3}dt'dx'da'\zeta_\varepsilon(t-t',x-x',a-a',p-p^\pm_1(t',x',a'))$$
$$\cdot f^\pm_{\varepsilon,2}(t,z,a)\psi_R(x,p,a)[\partial_x p^\pm_1(t',x',a')(\mathcal{F}_{x,1}-\mathcal{F}_{x,2})(t,x,p^\pm_1(t',x',a'))$$
$$+(\mathcal{F}_{p,1}-\mathcal{F}_{p,2})(t,x,0)-(\mathcal{F}_{p,1}-\mathcal{F}_{p,2})(t,x,p^\pm_1(t',x',a'))]$$

$$
\begin{aligned}
&\leq 2\int_0^T \||\partial_x p_1^\pm| \overset{txa}{*} \zeta_{\varepsilon_{txa}}\|_{L^1(\mathcal{D})}\|\mathcal{F}_{x,1}-\mathcal{F}_{x,2}\|_{L^\infty(\mathfrak{D})}dt \\
&\quad + 4\int_0^T dt\int_{\mathcal{D}} dxda \sup_{\xi\in\mathbb{R}}|\mathcal{F}_{p,1}-\mathcal{F}_{p,2}|(t,x,\xi) \\
&\leq 2\|p_1^\pm\|_{L^\infty(0,T,\mathrm{BV}(\mathcal{D}))}\int_0^T \|\mathcal{F}_{x,1}-\mathcal{F}_{x,2}\|_{L^\infty(\mathfrak{D})}dt \\
&\quad + 4\int_0^T dt\int_{\mathbb{T}_L} dx \sup_{\xi\in\mathbb{R}}|\mathcal{F}_{p,1}-\mathcal{F}_{p,2}|.
\end{aligned}
\tag{3.82}
$$

In the above we have used the fact that

$$
\begin{aligned}
\||\partial_x p_1^\pm| \overset{txa}{*} \zeta_{\varepsilon_{txa}}\|_{L^\infty(0,T;L^1(\mathcal{D}))} &\leq \||\partial_x p_1^\pm| \overset{xa}{*} \zeta_{\varepsilon_{xa}}\|_{L^\infty(0,T;L^1(\mathcal{D}))} \\
&\leq \|p_1^\pm\|_{L^\infty(0,T,\mathrm{BV}(\mathcal{D}))}.
\end{aligned}
$$

Using d'Alembert integral representation formula, and a Gronwall lemma we obtain

$$
\begin{aligned}
&\|\mathcal{F}_{x,1}-\mathcal{F}_{x,2}\|_{L^\infty(\mathfrak{D})} \\
&\quad\leq \|\mathbf{A}_{\perp,1}-\mathbf{A}_{\perp,2}\|_{\mathbb{L}^\infty(\mathbb{T}_L)} \\
&\quad\leq \|\mathbf{A}^0_{\perp,1}-\mathbf{A}^0_{\perp,2}\|_{\mathbb{L}^\infty(\mathbb{T}_L)} + t\|\mathbf{A}^1_{\perp,1}-\mathbf{A}^1_{\perp,2}\|_{\mathbb{L}^\infty(\mathbb{T}_L)} \\
&\qquad + \frac{1}{2}\left\lceil\frac{2t}{L}\right\rceil\int_0^t d\tau(\|\mathbf{A}_{\perp,1}-\mathbf{A}_{\perp,2}\|_{\mathbb{L}^\infty(\mathbb{T}_L)}\|\rho_\gamma[\mathbf{p}_1]\|_{L^1(\mathbb{T}_L)} \\
&\qquad + \|(\rho_\gamma[\mathbf{p}_1]-\rho_\gamma[\mathbf{p}_2])\mathbf{A}_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}) \\
&\quad\leq e^{\frac{t}{2}\left\lceil\frac{2t}{L}\right\rceil\left(\|p_1^-\|_{L^\infty(0,T,L^1(\mathcal{D}))}+\|p_1^+\|_{L^\infty(0,T,L^1(\mathcal{D}))}+\pi\|\mathbf{A}_{\perp,2}\|_{L^\infty(0,T,\mathbb{L}^1(\mathbb{T}_L))}\right)} \\
&\qquad \cdot\left\{\|\mathbf{A}^0_{\perp,1}-\mathbf{A}^0_{\perp,2}\|_{\mathbb{L}^\infty(\mathbb{T}_L)} + t\|\mathbf{A}^1_{\perp,1}-\mathbf{A}^1_{\perp,2}\|_{\mathbb{L}^\infty(\mathbb{T}_L)} + \frac{\pi}{2}\left\lceil\frac{2t}{L}\right\rceil\right. \\
&\qquad \left.\cdot\ \|\mathbf{A}_{\perp,2}\|_{\mathbb{L}^\infty(\Omega)}\int_0^t d\tau\ \left(\|p_1^- - p_2^-\|_{L^1(\mathcal{D})} + \|p_1^+ - p_2^+\|_{L^1(\mathcal{D})}\right)\right\}.
\end{aligned}
\tag{3.83}
$$

Using (3.12) and (3.13), we obtain

$$
\begin{aligned}
&\int_{\mathbb{T}_L} dx \sup_{\xi\in\mathbb{R}}|\mathcal{F}_{p,1}-\mathcal{F}_{p,2}| \\
&\quad\leq \|K\|_{L^1(\mathbb{T}_L)\times L^\infty(\mathbb{T}_L)}(\|p_1^- - p_2^-\|_{L^1(\mathcal{D})} + \|p_1^+ - p_2^+\|_{L^1(\mathcal{D})}) \\
&\qquad + (\|\partial_x\mathbf{A}_{\perp,1}\|_{\mathbb{L}^\infty(\Omega)} + \|\partial_x\mathbf{A}_{\perp,2}\|_{\mathbb{L}^\infty(\Omega)})\|\mathbf{A}_{\perp,1}-\mathbf{A}_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)} \\
&\qquad + \|\mathbf{A}_{\perp,2}\|_{\mathbb{L}^\infty(\Omega)}\|\partial_x\mathbf{A}_{\perp,1}-\partial_x\mathbf{A}_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|K\|_{L^1(\mathbb{T}_L)\times L^\infty(\mathbb{T}_L)}(\|p_1^- - p_2^-\|_{L^1(\mathcal{D})} + \|p_1^+ - p_2^+\|_{L^1(\mathcal{D})})\\
&\quad + \mathcal{C}^\sharp(t, \|p_i^\pm\|_{L^\infty(Q)}, \|\mathbf{A}_{\perp,i}\|_{\mathbb{L}^\infty(\Omega)}, \|\partial_x \mathbf{A}_{\perp,i}\|_{\mathbb{L}^\infty(\Omega)})\\
&\quad \cdot \Bigg\{ \|\mathbf{A}^1_{\perp,1} - \mathbf{A}^1_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)} + \|\mathbf{A}^0_{\perp,1} - \mathbf{A}^0_{\perp,2}\|_{\mathbb{W}^{1,1}(\mathbb{T}_L)}\\
&\quad + \int_0^t d\tau\,(\|p_1^- - p_2^-\|_{L^1(\mathcal{D})} + \|p_1^+ - p_2^+\|_{L^1(\mathcal{D})})\Bigg\}.
\end{aligned} \tag{3.84}
$$

From estimates (3.73) to (3.84) we get

$$
\begin{aligned}
&\|p_1^-(T) - p_2^-(T)\|_{L^1(\mathcal{D})} + \|p_1^+(T) - p_2^+(T)\|_{L^1(\mathcal{D})}\\
&\quad \leq \mathcal{C}^\star(T, L, \|p_i^\pm\|_{L^\infty(Q)}, \|p_i^\pm\|_{L^\infty(0,T;\mathrm{BV}(\mathcal{D}))}, \|\mathbf{A}_{\perp,i}\|_{\mathbb{L}^\infty(\Omega)}, \|\partial_x \mathbf{A}_{\perp,i}\|_{\mathbb{L}^\infty(\Omega)})\\
&\quad \cdot \Bigg\{ \|\mathbf{A}^0_{\perp,1} - \mathbf{A}^0_{\perp,2}\|_{\mathbb{L}^1\cap\mathbb{L}^\infty(\mathbb{T}_L)} + \|\mathbf{A}^1_{\perp,1} - \mathbf{A}^1_{\perp,2}\|_{\mathbb{L}^1\cap\mathbb{L}^\infty(\mathbb{T}_L)}\\
&\quad + \|\mathbf{A}^{0\prime}_{\perp,1} - \mathbf{A}^{0\prime}_{\perp,2}\|_{\mathbb{L}^1(\mathbb{T}_L)} + \int_0^T dt\,(\|p_1^- - p_2^-\|_{L^1(\mathcal{D})} + \|p_1^+ - p_2^+\|_{L^1(\mathcal{D})})\Bigg\}\\
&\quad + \|p_{0\,1}^- - p_{0\,2}^-\|_{L^1(\mathcal{D})} + \|p_{0\,1}^+ - p_{0\,2}^+\|_{L^1(\mathcal{D})},
\end{aligned}
$$

which ends the proof by using once again a Gronwall lemma. □

3.6. *Return to the relativistic Vlasov–Maxwell equations*

The global weak entropy solutions of the relativistic waterbag continuum (2.9)–(2.12), are linked to special class of weak solutions of the relativistic Vlasov–Maxwell equations with kinetic entropy defect measure as follows.

Theorem 3.8. *The system* (2.9)–(2.12) *is equivalent to the relativistic Vlasov–Maxwell*

$$\partial_t f + \nabla_{x,p} \cdot (\mathcal{F} f) = \partial_p m, \tag{3.85}$$

with kinetic entropy defect measure m defined by

$$m(t,x,p) = \int_0^1 (m^+(t,x,a,p) - m^-(t,x,p,a))da,$$

and where

$$
\begin{gathered}
f(t,x,p) = \int_0^1 (\chi(p,p^+) - \chi(p,p^-))da,\\
\mathcal{F}_x = \partial_p \mathcal{H}, \quad \mathcal{F}_p = -\partial_x \mathcal{H}, \quad \mathcal{H} = \sqrt{1 + p^2 + |\mathbf{A}_\perp|^2} + \phi - 1.
\end{gathered} \tag{3.86}
$$

In the above equations the waterbag continuum $p^\pm$ are the unique weak entropy solutions of the system (2.9)–(2.12) *given by Theorems* 3.4 *and* 3.7, *or equivalently* *are such that the kinetic functions* $\chi(p, p^{\pm})$ *and the kinetic entropy defect measures* $m^{\pm}$ *satisfy the kinetic equation* (3.42) *of Theorem* 3.6 *in* $\mathscr{D}'(\Sigma)$. *The Vlasov equation* (3.85) *is coupled to electromagnetic field equations*

$$(\partial_t^2 - \partial_x^2 + \rho_\gamma)\mathbf{A}_\perp = 0, \quad -\partial_x^2\phi = \rho - 1, \quad \partial_t E_x + J_x = \frac{1}{L}\int_{\mathbb{T}_L} J_x(t,x)dx,$$
$$E_x = -\partial_x\phi,$$

where the charge and current densities are defined by

$$\rho_\gamma = \int_{\mathbb{R}} \frac{f(t,x,p)}{\sqrt{1+p^2+|\mathbf{A}_\perp|^2}}dp, \quad J_x = \int_{\mathbb{R}} \frac{pf(t,x,p)}{\sqrt{1+p^2+|\mathbf{A}_\perp|^2}}dp, \quad \rho = \int_{\mathbb{R}} f(t,x,p)dp.$$

Moreover, the mass

$$\mathfrak{M}(t) = \int_{\mathbb{T}_L} \rho dx$$

is preserved, while the total energy

$$\mathcal{E}(t) = \int_{\mathfrak{D}} f(\gamma-1)dpdx + \frac{1}{2}\int_{\mathbb{T}_L} (|\partial_t\mathbf{A}_\perp|^2 + |\partial_x\mathbf{A}_\perp|^2 + |\partial_x\phi|^2)dx$$

is bounded.

Proof. After subtracting Eqs. (3.42) and integrating over a we obtain Eq. (3.85). Since the kinetic entropy defect measure m is compactly supported in the momentum variable p, after integrating Eq. (3.85) with respect to the momentum variable p, we get $\frac{d}{dt}\mathfrak{M}(t) = 0$. Multiplying Eq. (3.85) by $(\gamma - 1)$, and integrating with respect to the variables (x,p), we obtain

$$\partial_t\left(\int_{\mathfrak{D}}(\gamma-1)fdxdp\right) - \int_{\mathfrak{D}} f\partial_t\gamma dxdp$$
$$+\int_{\mathfrak{D}}\nabla\cdot((\gamma-1)\mathcal{F}f)dxdp - \int_{\mathfrak{D}}\mathcal{F}\cdot\nabla\gamma f dxdp = \int_{\mathfrak{D}}(\gamma-1)\partial_p m.$$

Since $\mathcal{F}\cdot\nabla\gamma = -\partial_x\phi\partial_p\gamma$ and $\partial_t\gamma = \partial_t(|\mathbf{A}_\perp|^2/2)/\gamma$, using Maxwell equations we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathfrak{D}}(|\partial_t\mathbf{A}_\perp|^2 + |\partial_x\mathbf{A}_\perp|^2)dx = -\frac{1}{2}\int_{\mathbb{T}_L}\rho_\gamma\frac{d}{dt}|\mathbf{A}_\perp|^2$$

and the Ampère equation, we get $J_x\partial_x\phi = \frac{1}{2}\frac{d}{dt}|\partial_x\phi|^2$, so that we finally obtain

$$\frac{d}{dt}\left(\int_{\mathfrak{D}} f(\gamma-1)dpdx + \frac{1}{2}\int_{\mathbb{T}_L}(|\partial_t\mathbf{A}_\perp|^2 + |\partial_x\mathbf{A}_\perp|^2 + |\partial_x\phi|^2)dx\right)$$
$$= -\int_{\mathfrak{D}} m\frac{p}{\gamma}dpdx. \tag{3.87}$$

Integrating in time (3.87), and using the notation $\|p_0\|_{L^2(\mathcal{D})} = \|p_0^-\|_{L^2(\mathcal{D})} + \|p_0^+\|_{L^2(\mathcal{D})}$, we obtain

$$\begin{aligned}\mathcal{E}(T) - \mathcal{E}(0) &\leq \int_\Sigma (m^- + m^+) dtdxdadp \\ &\leq \frac{1}{2}\|p_0\|_{L^2(\mathcal{D})} - \int_\Sigma (\partial_x \mathcal{H})(t,x,p)(\chi(p,p^-) + \chi(p,p^+))dtdxdadp \\ &\leq \frac{1}{2}\|p_0\|_{L^2(\mathcal{D})} + (\|\partial_x \mathbf{A}_\perp\|_{\mathbb{L}^\infty(\Omega)} + \|\partial_x\phi\|_{L^\infty(\Omega)}) \\ &\quad \cdot (\|p^-\|_{L^1(Q)} + \|p^+\|_{L^1(Q)}) < \infty,\end{aligned}$$

which ends the proof. □

Remark 3.4. The right-hand side of the relativistic Vlasov–Maxwell equation (3.85), or in other words the kinetic entropy defect measure m, can be interpreted as a Lagrange multiplier associated to the constraint that the distribution function keeps the special shape of a waterbag decomposition (3.86).

Appendix A. Proof of Theorem 3.1

The global existence follows from a Banach fixed point theorem which is based on the continuity and contraction properties of a map that we will define further. We first make the change of unknowns $p^\pm(t,x,a) = q^\pm(t,x,a)\exp(\Lambda t)$ with $\Lambda > 0$. Therefore the unknowns $q^\pm$ satisfy the following equations

$$\partial_t q^\pm - \varepsilon\partial_x^2 q^\pm + \Lambda q^\pm = -e^{-\Lambda t}\partial_x \mathcal{H}^\pm(\mathbf{q}e^{\Lambda t}), \tag{A.1}$$

where we have used the notation $\mathbf{q} = (q^-, q^+)$ to emphasize the fact that $\mathbf{p} \to \mathcal{H}^\pm(\mathbf{p})$ with $\mathcal{H}^\pm(\mathbf{p}) = \sqrt{1 + {p^\pm}^2 + |\mathbf{A}_\perp[\mathbf{p}]|^2} - 1 + \phi[\mathbf{p}] = \gamma^\pm(\mathbf{p}) + \phi[\mathbf{p}] - 1$, define maps which are continuous from $L^\infty(0,T;\mathbb{L}^2(\mathcal{D}))$ into $L^\infty(0,T;L^2(\mathcal{D}))$ as we will prove it below. Let us assume $q^\pm \in L^2(0,T;L^2(\mathcal{D}))$ for all fixed $T > 0$. We now consider the problem

$$\partial_t r^\pm - \varepsilon\partial_x^2 r^\pm + \Lambda r^\pm = f^\pm := -e^{-\Lambda t}\partial_x \mathcal{H}^\pm(\mathbf{q}e^{\Lambda t}), \tag{A.2}$$

$$r^\pm(t=0,x,a) = r_0^\pm(x,a). \tag{A.3}$$

Before going further, let us define the functional space $\mathbb{W}(0,T) = \{\varphi \in L^2(0,T;\mathbb{V}); \partial_t\varphi \in L^2(0,T;\mathbb{V}')\}$, with $\mathbb{V}$ and its dual $\mathbb{V}'$ respectively defined in the same way of the spaces (3.2) and (3.3), where the spaces $L^2(\mathcal{D})$, $H^1(\mathbb{T}_L)$ and $H^{-1}(\mathbb{T}_L)$ are respectively replaced by their vector-valued counterparts $\mathbb{L}^2(\mathcal{D})$, $\mathbb{H}^1(\mathbb{T}_L)$ and $\mathbb{H}^{-1}(\mathbb{T}_L)$. In the following $\mathfrak{p}^\alpha_{L,i}(t)$, $i \in \mathbb{N}^*$, denote polynomials in time of degree less than or equal to $\alpha \in \mathbb{R}^+$ and $\lceil r \rceil$ stands for the smallest integer greater than r.

Let us now show that for all $T > 0$, $f^{\pm} \in L^2(0,T;V')$. Obviously we have

$$\|\mathcal{H}^{\pm}(\mathbf{q}e^{\Lambda t})e^{-\Lambda t}\|_{L^2(Q)} \leq \|\mathbf{q}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{D}))} + \||\mathbf{A}_{\perp}[\mathbf{q}e^{\Lambda t}]|e^{-\Lambda t}\|_{L^2(\Omega)} + \|\phi[\mathbf{q}]\|_{L^2(\Omega)}. \tag{A.4}$$

Using d'Alembert integral representation formula, we obtain

$$\||\mathbf{A}_{\perp}[\mathbf{q}e^{\Lambda t}]|e^{-\Lambda t}\|_{L^2(\Omega)} \leq \frac{1}{(2\Lambda)^{1/2}}\|\mathbf{A}_{\perp}^0\|_{\mathbb{L}^2(\mathbb{T}_L)} + \left(\frac{L}{2}\left\lceil\frac{2T}{L}\right\rceil\right)^{1/2}\frac{1}{2\Lambda}\|\mathbf{A}_{\perp}^1\|_{\mathbb{L}^2(\mathbb{T}_L)} + \left(L\left\lceil\frac{2T}{L}\right\rceil\frac{1}{2\Lambda}\right)^{1/2}\frac{T}{2}(\|q^-\|_{L^2(Q)} + \|q^+\|_{L^2(Q)}). \tag{A.5}$$

Using (A.4) and (A.5), we obtain that

$$\|\mathcal{H}^{\pm}(\mathbf{q}e^{\Lambda t})e^{-\Lambda t}\|_{L^2(Q)} \leq C(T,L,\Lambda,\|G\|_{L^2(\mathbb{T}_L\times\mathbb{T}_L)},\|\mathbf{A}_{\perp}^0\|_{\mathbb{L}^2(\mathbb{T}_L)},\|\mathbf{A}_{\perp}^1\|_{\mathbb{L}^2(\Omega)}) \cdot(\|q^-\|_{L^2(Q)} + \|q^+\|_{L^2(Q)}), \tag{A.6}$$

where $G \in W^{1,\infty}(\mathbb{T}_L^2)$ is the Green function (or the fundamental solution) of the one-dimensional Laplace operator with periodic boundary conditions. Hence, for all $T > 0$, for all $\Lambda > 0$, $f^{\pm} \in L^2(0,T;V')$. Therefore using Theorem 4.1 of Chap. 3 and Theorem 3.1 of Chap. 1 in Ref. 52, we can show that the problem (A.2)–(A.3) has a unique solution $r^{\pm} \in \mathcal{W}(0,T) \cap \mathscr{C}(0,T;L^2(\mathcal{D}))$. Let $\mathcal{F}_{\Lambda} : L^2(0,T;\mathbb{L}^2(\mathcal{D})) \to \mathbb{W}(0,T) \cap \mathscr{C}(0,T;\mathbb{L}^2(\mathcal{D}))$ be the map defined by $\mathbf{r} = \mathcal{F}_{\Lambda}(\mathbf{q})$. Thus we can show that $\mathcal{F}_{\Lambda}$ is a contractive maps in $L^2(0,T;\mathbb{L}^2(\mathcal{D}))$ for Λ large enough. Let $\mathbf{q}_i \in \mathbb{L}^2(Q)$ and $\mathbf{r}_i = \mathcal{F}_{\Lambda}(\mathbf{q}_i) \in \mathbb{W}(0,T) \cap \mathscr{C}(0,T;\mathbb{L}^2(\mathcal{D}))$ for $i = 1,2$. We set $\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Therefore for all $w \in L^2(0,T;V)$, we have

$$\langle \partial_t r^{\pm} - \varepsilon\partial_x^2 r^{\pm} + \Lambda r^{\pm}, w\rangle = e^{-\Lambda t}\langle \mathcal{H}^{\pm}(\mathbf{q}_1 e^{\Lambda t}) - \mathcal{H}^{\pm}(\mathbf{q}_2 e^{\Lambda t}), \partial_x w\rangle. \tag{A.7}$$

If we set $w = r^{\pm}$, using the Young inequality $xy \leq \varepsilon x^2 + \frac{1}{4\varepsilon}y^2$, Eq. (A.7) becomes

$$\frac{1}{2}\|r^{\pm}(t)\|_{L^2(\mathcal{D})}^2 + \Lambda\int_0^t \|r^{\pm}(\tau)\|_{L^2(\mathcal{D})}^2 d\tau \leq \frac{1}{4\varepsilon}\int_0^t \|e^{-\Lambda\tau}(\mathcal{H}^{\pm}(\mathbf{q}_1 e^{\Lambda\tau}) - \mathcal{H}^{\pm}(\mathbf{q}_2 e^{\Lambda\tau}))\|_{L^2(\mathcal{D})}^2 d\tau. \tag{A.8}$$

Let us estimate the right-hand side Eq. (A.8). Using d'Alembert integral representation formula we obtain

$$\||\mathbf{A}_{\perp}[\mathbf{q}_1 e^{\Lambda t}] - \mathbf{A}_{\perp}[\mathbf{q}_2 e^{\Lambda t}]|\|_{L^2(\mathbb{T}_L)} \leq \mathfrak{p}_{L,1}^1(t)\int_0^t \{\||\mathbf{A}_{\perp}[\mathbf{q}_1 e^{\Lambda\tau}] - \mathbf{A}_{\perp}[\mathbf{q}_2 e^{\Lambda\tau}]|\|_{L^2(\mathbb{T}_L)}\|\rho_{\gamma}[\mathbf{q}_1 e^{\Lambda\tau}]\|_{L^2(\mathbb{T}_L)} + \||(\rho_{\gamma}[\mathbf{q}_1 e^{\Lambda\tau}] - \rho_{\gamma}[\mathbf{q}_2 e^{\Lambda\tau}])\mathbf{A}_{\perp}[\mathbf{q}_2 e^{\Lambda\tau}]|\|_{L^1(\mathbb{T}_L)}\}d\tau, \tag{A.9}$$

where $\mathfrak{p}^1_{L,1}(t) = \frac{1}{2}\lceil \frac{2t}{L} \rceil L^{1/2}$. Let us first estimate the term $\rho_\gamma[\mathbf{q}_i e^{\Lambda\tau}]$ in (A.9). Using the change of variable $w = pe^{-\Lambda\tau}$ we obtain

$$\|\rho_\gamma[\mathbf{q}_i e^{\Lambda\tau}]\|_{L^2(\mathbb{T}_L)} \leq 2(\|q_i^-\|_{L^2(Q)} + \|q_i^+\|_{L^2(Q)} + L^{1/2}). \tag{A.10}$$

Let us now estimate the second term of the right-hand side of (A.9). Using definition (2.14) and obvious estimates we obtain

$$\begin{aligned} &\||(\rho_\gamma[\mathbf{q}_1 e^{\Lambda\tau}] - \rho_\gamma[\mathbf{q}_2 e^{\Lambda\tau}])\mathbf{A}_\perp[\mathbf{q}_2 e^{\Lambda\tau}]|\|_{L^1(\mathbb{T}_L)} \\ &\quad \leq 2\||\mathbf{A}_\perp[\mathbf{q}_1 e^{\Lambda t}] - \mathbf{A}_\perp[\mathbf{q}_2 e^{\Lambda t}]|\|_{L^2(\mathbb{T}_L)}(\|q_1^-\|_{L^\infty(0,T;L^2(\mathcal{D}))} + \|q_1^+\|_{L^\infty(0,T;L^2(\mathcal{D}))} \\ &\qquad + L^{1/2}) + e^{\Lambda\tau}L^{1/2}(\|q_1^- - q_2^-\|_{L^2(Q)} + \|q_1^+ - q_2^+\|_{L^2(Q)}). \end{aligned} \tag{A.11}$$

Substituting (A.10) and (A.11) into (A.9) and using a Gronwall lemma we obtain

$$\begin{aligned} &\||\mathbf{A}_\perp[\mathbf{q}_1 e^{\Lambda t}] - \mathbf{A}_\perp[\mathbf{q}_2 e^{\Lambda t}]|e^{-\Lambda t}\|_{L^2(\Omega)} \\ &\quad \leq \left(\frac{TL}{2\Lambda}\right)^{1/2} \mathfrak{p}^1_{L,1}(T) e^{4T\mathfrak{p}^1_{L,1}(T)(\lambda + L^{1/2})}(\|q_1^- - q_2^-\|_{L^2(Q)} + \|q_1^+ - q_2^+\|_{L^2(Q)}), \end{aligned} \tag{A.12}$$

where $\lambda := \|q_1^-\|_{L^\infty(0,T;L^2(\mathcal{D}))} + \|q_1^+\|_{L^\infty(0,T;L^2(\mathcal{D}))}$. Therefore using (A.8), (A.12) and by noting that

$$\begin{aligned} &\|(\mathcal{H}^\pm(\mathbf{q}_1 e^{\Lambda t}) - \mathcal{H}^\pm(\mathbf{q}_2 e^{\Lambda t}))e^{-\Lambda t}\|_{L^2(Q)} \leq \|\mathbf{q}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{D}))} \\ &\quad + \||\mathbf{A}_\perp[\mathbf{q}_1 e^{\Lambda t}] - \mathbf{A}_\perp[\mathbf{q}_2 e^{\Lambda t}]|e^{-\Lambda t}\|_{L^2(\Omega)} + \|\phi[\mathbf{q}_2] - \phi[\mathbf{q}_1]\|_{L^2(\Omega)}, \end{aligned}$$

we obtain

$$\|r^-\|_{L^2(Q)} + \|r^+\|_{L^2(Q)} \leq \frac{C_T(T, L, \lambda, \|G\|_{L^2(\mathbb{T}_L\times\mathbb{T}_L)})}{(\varepsilon\Lambda)^{1/2}}(\|q^-\|_{L^2(Q)} + \|q^+\|_{L^2(Q)}),$$

where

$$\begin{aligned} &C_T(T, L, \lambda, \|G\|_{L^2(\mathbb{T}_L\times\mathbb{T}_L)}) \\ &\quad := 1 + \|G\|_{L^2(\mathbb{T}_L\times\mathbb{T}_L)} + \left(\frac{TL}{2\Lambda}\right)^{1/2} \mathfrak{p}^1_{L,1}(T) e^{4T\mathfrak{p}^1_{L,1}(T)(\lambda + L^{1/2})}. \end{aligned} \tag{A.13}$$

Therefore for all final time $T > 0$, there exists $\Lambda_T \in \mathbb{R}^+$ such that the map $\mathcal{F}_{\Lambda_T} : L^2(0,T;\mathbb{L}^2(\mathcal{D})) \to L^2(0,T;\mathbb{L}^2(\mathcal{D}))$ is a contraction in $L^2(0,T;\mathbb{L}^2(\mathcal{D}))$ and has a unique fixed point $\mathbf{q} \in \mathbb{W}(0,T) \cap \mathscr{C}(0,T;\mathbb{L}^2(\mathcal{D}))$ which satisfies the system (A.1). Finally since $\mathbf{p} = \mathbf{q}\exp(\Lambda t) \in \mathbb{W}(0,T) \cap \mathscr{C}(0,T;\mathbb{L}^2(\mathcal{D}))$, Eqs. (3.1) and (2.10)–(2.12) has a unique global strong solution. Obviously the solution satisfies the regularity properties (3.5).

Appendix B. Proof of Theorem 3.2

Let us define the cutoff function $\Theta(r) \in \mathscr{C}_0^\infty(\mathbb{R})$ such that $\Theta(r) = 1$ if $r \leq \mathcal{R}$, and $\Theta(r) = 0$ if $r \geq \mathcal{R}+1$, with $\mathcal{R} := \max\{\|p_0^-\|_{L^\infty(\mathcal{D})}, \|p_0^+\|_{L^\infty(\mathcal{D})}\} + T(\|\partial_x\widetilde{\phi}\|_{L^\infty(\Omega)} + \|\partial_x\widetilde{\mathbf{A}}_\perp\|_{\mathbb{L}^\infty(\Omega)})$, and where the tilde notation is explained below. We define $\widetilde{\mathcal{H}}^\pm(\mathbf{p}) = \Theta(p^\pm)\mathcal{H}^\pm(\mathbf{p})$. Using integral formulation of the waves Eq. (2.10) and Poisson equation (2.11), by a Gronwall lemma we can show that the maps $\mathbf{p} \to \mathcal{H}^\pm(\mathbf{p})$ are Lipschitz continuous from $L^\infty(0,T;\mathbb{L}^2(\mathcal{D}))$ into $L^\infty(0,T;L^2(\mathcal{D}))$, i.e. there exists a constant $C_T := C_T(T,L,\lambda,\|G\|_{L^2(\mathbb{T}_L\times\mathbb{T}_L)})$ where $\lambda := \lambda(\|p_1^-\|_{L^\infty(0,T;L^2(\mathcal{D}))}, \|p_1^+\|_{L^\infty(0,T;L^2(\mathcal{D}))}, \|p_2^-\|_{L^\infty(0,T;L^2(\mathcal{D}))}, \|p_2^+\|_{L^\infty(0,T;L^2(\mathcal{D}))})$, and $G \in W^{1,\infty}(\mathbb{T}_L^2)$ is the fundamental solution of the one-dimensional Laplace operator with periodic boundary conditions, such that $\|\mathcal{H}^\pm(\mathbf{p}_1) - \mathcal{H}^\pm(\mathbf{p}_2)\|_{L^\infty(0,T;L^2(\mathcal{D}))} \leq C_T\|\mathbf{p}_1 - \mathbf{p}_2\|_{L^\infty(0,T;\mathbb{L}^2(\mathcal{D}))}$, for all $\mathbf{p}_i \in L^\infty(0,T;\mathbb{L}^2(\mathcal{D}))$ with $i = 1,2$. Since $\mathcal{H}^\pm$ are Lipschitz in $\mathbf{p}$, we can easily proof that $\widetilde{\mathcal{H}}^\pm$ are also Lipschitz in $\mathbf{p}$. Following the proof of Theorem 3.1, we can show that there exists a unique solution $\widetilde{p}^\pm \in \mathscr{C}(0,T;L^2(\mathcal{D})) \cap \mathcal{W}(0,T)$ to the problem

$$\partial_t\widetilde{p}^\pm + \partial_x^2\widetilde{\mathcal{H}}^\pm(\widetilde{\mathbf{p}}) = \varepsilon\partial_x^2\widetilde{p}^\pm, \quad \widetilde{p}^\pm(t=0) = \widetilde{p}_0^\pm, \tag{B.1}$$

$$\partial_t^2\widetilde{\mathbf{A}}_\perp - \partial_x^2\widetilde{\mathbf{A}}_\perp = -\widetilde{\mathbf{A}}_\perp\widetilde{\rho}_\gamma, \quad \widetilde{\mathbf{A}}_\perp(t=0) = \mathbf{A}_\perp^0, \quad \partial_t\widetilde{\mathbf{A}}_\perp(t=0) = \mathbf{A}_\perp^1, \tag{B.2}$$

$$\widetilde{E}_x = -\partial_x\widetilde{\phi}, \quad -\partial_x^2\widetilde{\phi} = \widetilde{\rho} - 1, \quad \partial_t\widetilde{E}_x = -\widetilde{J}_x + \frac{1}{L}\int_{\mathbb{T}_L}\widetilde{J}_x(t,x)dx. \tag{B.3}$$

From d'Alembert integral representation formula we obtain

$$\|\partial_x\widetilde{\mathbf{A}}_\perp\|_{\mathbb{L}^\infty(\Omega)} \leq \|\widetilde{\mathbf{A}}_\perp^{0\prime}\|_{\mathbb{L}^\infty(\mathbb{T}_L)} + \|\widetilde{\mathbf{A}}_\perp^1\|_{\mathbb{L}^\infty(\mathbb{T}_L)} + \frac{1}{2}\left(T\left\lceil\frac{T}{L}\right\rceil\right)^{1/2}(\|\widetilde{p}^-\|_{L^\infty(0,T;L^2(\mathcal{D}))} + \|\widetilde{p}^+\|_{L^\infty(0,T;L^2(\mathcal{D}))}) \tag{B.4}$$

and thus $\partial_x\widetilde{\mathbf{A}}_\perp \in \mathbb{L}^\infty(\Omega)$. Using equation

$$E_x(t,x) = \int_{\mathbb{T}_L} K(x,y)(\rho-1)dy,$$

where $K(x,y) = -\partial_x G(x,y)$ and from the Sobolev embedding $H^2(\mathbb{T}_L) \hookrightarrow W^{1,\infty}(\mathbb{T}_L)$ we get $\partial_x\widetilde{\phi} \in L^\infty(\Omega)$.

Let us now show that the solution $\{\widetilde{p}^\pm, \widetilde{\phi}, \widetilde{\mathbf{A}}_\perp\}$ of the system (B.1)–(B.3) is in fact the solution of the problem (3.1) and (2.10)–(2.12). To this aim, we need to verify that the solution $\{\widetilde{p}^\pm, \widetilde{\phi}, \widetilde{\mathbf{A}}_\perp\}$ satisfies the maximum principle (3.6), i.e.

$$\|\widetilde{p}^\pm(t)\|_{L^\infty(\mathcal{D})} \leq \|p_0^\pm\|_{L^\infty(\mathcal{D})} + \int_0^t d\tau\{\|\partial_x\widetilde{\phi}(\tau)\|_{L^\infty(\mathbb{T}_L)} + \|\partial_x\widetilde{\mathbf{A}}_\perp(\tau)\|_{\mathbb{L}^\infty(\mathbb{T}_L)}\}, \tag{B.5}$$

so that $\Theta \equiv 1$ and $\widetilde{\mathcal{H}}^{\pm}(\widetilde{\mathbf{p}}) = \mathcal{H}^{\pm}(\widetilde{\mathbf{p}})$. Let us define

$$q^{\pm} := \widetilde{p}^{\pm} - \|p_0^{\pm}\|_{L^\infty(\mathcal{D})} - \int_0^t d\tau \{\|\partial_x \widetilde{\phi}(\tau)\|_{L^\infty(\mathbb{T}_L)} + \|\partial_x \widetilde{\mathbf{A}}_\perp(\tau)\|_{\mathbb{L}^\infty(\mathbb{T}_L)}\}.$$

We have to show that $q_+^{\pm} = (q^{\pm})_+ = \max\{q^{\pm}, 0\} = 0$. Since $|(x - x_0)_+| \le |x|$, for $x_0 \ge 0$, we have $|q_+^{\pm}| \le |\widetilde{p}^{\pm}|$, so that $q_+^{\pm}(t) \in L^2(\mathcal{D})$ for all time t. Moreover, $\widetilde{p}^{\pm}(t) \in V$ for almost every time t and $\partial_x q^{\pm} = \partial_x \widetilde{p}^{\pm}$, hence $q^{\pm}(t) \in V$ for almost every time t. Consequently, since the function $(\cdot)_+$ is a Lipschitz continuous function such that $(\cdot)'_+ = \mathrm{sign}(\cdot)$, $q^{\pm}(t) \in V$ implies $q_+^{\pm}(t) \in V$ by the chain rule formula. Therefore, using (B.1) we obtain

$$\partial_t q^{\pm} + \|\partial_x \widetilde{\phi}(t)\|_{L^\infty(\mathbb{T}_L)} + \|\partial_x \widetilde{\mathbf{A}}_\perp(t)\|_{\mathbb{L}^\infty(\mathbb{T}_L)} - \varepsilon \partial_x^2 q^{\pm} = \partial_x \widetilde{\mathcal{H}}^{\pm}(\widetilde{\mathbf{p}}).$$

Then for all $\psi \in V$ we have

$$\begin{aligned}
&\langle \partial_t q^{\pm}, \psi\rangle + \langle \|\partial_x \widetilde{\phi}(t)\|_{L^\infty(\mathbb{T}_L)} + \|\partial_x \widetilde{\mathbf{A}}_\perp(t)\|_{\mathbb{L}^\infty(\mathbb{T}_L)}, \psi\rangle + \varepsilon\langle \partial_x q^{\pm}, \partial_x \psi\rangle \\
&\quad = \int_0^1 da \int_{\mathbb{T}_L} dx\, \Theta'(\widetilde{p}^{\pm}) \widetilde{\mathcal{H}}^{\pm}(\widetilde{\mathbf{p}}) \partial_x \widetilde{p}^{\pm} \psi + \int_0^1 da \int_{\mathbb{T}_L} dx\, \Theta(\widetilde{p}^{\pm}) \partial_x \widetilde{\phi} \psi \\
&\qquad + \int_0^1 da \int_{\mathbb{T}_L} dx\, \Theta(\widetilde{p}^{\pm}) \left(\frac{\widetilde{p}^{\pm} \partial_x \widetilde{p}^{\pm}}{\gamma^{\pm}(\widetilde{\mathbf{p}})} + \frac{\widetilde{\mathbf{A}}_\perp \cdot \partial_x \widetilde{\mathbf{A}}_\perp}{\gamma^{\pm}(\widetilde{\mathbf{p}})} \right) \psi. \qquad \text{(B.6)}
\end{aligned}$$

If we take $\psi = q_+^{\pm}$ in (B.6), and observe (thanks to the chain rule formula $\partial_{z_i}(\Psi \circ u) = (\Psi' \circ u)\partial_{z_i} u$, for $i = 1, \dots, d$, with $\Psi \in W^{1,\infty}(\mathbb{R})$ and $u \in H^1(\mathbb{R}^d)$) that $q_+^{\pm} \partial_t q_+^{\pm} = q_+^{\pm} \partial_t q^{\pm}$, $|\partial_x q_+^{\pm}|^2 = \partial_x q^{\pm} \partial_x q_+^{\pm}$, $q_+^{\pm} \partial_x q_+^{\pm} = q_+^{\pm} \partial_x q^{\pm}$, $q_+^{\pm}(t = 0) = 0$, $\|\Theta\|_{L^\infty(\mathbb{R})} \le 1$, then using the Young inequality $xy \le \frac{\varepsilon}{2} x^2 + \frac{1}{2\varepsilon} y^2$, we get

$$\frac{1}{2}\|q_+^{\pm}\|^2_{L^2(\mathcal{D})} \le \frac{1}{2\varepsilon} \int_0^t d\tau \left\{ \|\Theta'(\widetilde{p}^{\pm}) \widetilde{\mathcal{H}}^{\pm}(\widetilde{\mathbf{p}}) q_+^{\pm}\|^2_{L^2(\mathcal{D})} + \left\| \Theta(\widetilde{p}^{\pm}) \frac{\widetilde{p}^{\pm}}{\gamma^{\pm}(\widetilde{\mathbf{p}})} q_+^{\pm} \right\|^2_{L^2(\mathcal{D})} \right\}. \qquad \text{(B.7)}$$

Since there exists a numerical constant C such that $\|\Theta'\|_{L^\infty(\mathbb{R})} \le C$, we deduce from (B.7) that

$$\begin{aligned}
\|q_+^{\pm}\|^2_{L^2(\mathcal{D})} &\le \frac{1}{2\varepsilon} \{1 + \mathcal{K}\|\Theta'\|^2_{L^\infty(\mathbb{R})} (1 + \mathcal{R}^2 + \|\widetilde{\mathbf{A}}_\perp\|^2_{\mathbb{L}^\infty(\Omega)} \\
&\quad + \|\widetilde{\phi}\|^2_{L^\infty(\Omega)})\} \int_0^t d\tau \|q_+^{\pm}\|^2_{L^2(\mathcal{D})}, \qquad \text{(B.8)}
\end{aligned}$$

where $\mathcal{K}$ is a pure numerical constant. Using a Gronwall lemma we deduce from (B.8) that $q_+^{\pm} = 0$, since $q_+^{\pm}(t = 0) = 0$. Similarly we can prove that

$$\left(-\widetilde{p}^{\pm}(t) - \|p_0^{\pm}\|_{L^\infty(\mathcal{D})} - \int_0^t d\tau \left\{ \|\partial_x \widetilde{\phi}(\tau)\|_{L^\infty(\mathbb{T}_L)} + \|\partial_x \widetilde{\mathbf{A}}_\perp(\tau)\|_{\mathbb{L}^\infty(\mathbb{T}_L)} \right\} \right)_+ = 0,$$

and consequently (B.5) holds for almost every time $t \in [0,T]$. Therefore $\widetilde{\mathcal{H}}^{\pm}(\widetilde{\mathbf{p}}) = \mathcal{H}^{\pm}(\widetilde{\mathbf{p}})$, and the solution $\{\widetilde{p}^{\pm}, \widetilde{\phi}, \widetilde{\mathbf{A}}_\perp\}$ of the problem (B.1)–(B.3) is in fact the solution of the problem (3.1) and (2.10)–(2.12) such that $\widetilde{p}^{\pm} \in \mathscr{C}(0,T;L^2(\mathcal{D})) \cap \mathcal{W}(0,T) \cap L^\infty(Q)$, $\widetilde{\phi} \in L^\infty(0,T;W^{1,\infty}(\mathbb{T}_L))$, $\widetilde{\mathbf{A}}_\perp \in L^\infty(0,T;\mathbb{W}^{1,\infty}(\mathbb{T}_L))$ and satisfy the maximum principle (B.5). Using d'Alembert integral representation formula, and similar estimates as (B.4) we obtain that $\partial_t \mathbf{A}_\perp \in \mathbb{L}^\infty(\Omega)$ which proves that $\mathbf{A}_\perp \in \mathbb{W}^{1,\infty}(\Omega)$. Since $\mathcal{H}^{\pm} \in L^\infty(Q)$, and using equation

$$\begin{aligned}\partial_t \phi(t,x) &= \int_{\mathbb{T}_L} dy \partial_y G(x,y) \int_0^1 da(\mathcal{H}(t,y,p^+) - \mathcal{H}(t,y,p^-)), \\ &= \int_{\mathbb{T}_L} \partial_y G(x,y) J_x(t,y) dy,\end{aligned}$$

we get $\partial_t \phi \in L^\infty(\Omega)$ which proves that $\phi \in W^{1,\infty}(\Omega)$. It remains to check the uniqueness of the solution $p^{\pm}$. Let $p_i^{\pm} \in \mathscr{C}(0,T;L^2(\mathcal{D})) \cap \mathcal{W}(0,T) \cap L^\infty(Q)$, $\phi_i \in W^{1,\infty}(\Omega)$, $\mathbf{A}_{\perp,\,i} \in \mathbb{W}^{1,\infty}(\Omega)$ with $i = 1,2$ two solutions of the problem (3.1) and (2.10)–(2.12). If we truncate the Hamiltonian $\mathcal{H}$ defined in (2.13) outside the interval $\mathcal{I}_{\max} = \left\{p \in \mathbb{R},\ |p| \le \max_{i\in\{1,2\}} \max_{\alpha\in\{-,+\}} \|p_i^\alpha\|_{L^\infty(Q)}\right\}$, we obtain a Lipschitz function $\mathcal{H}$ in $\mathbf{p}$ and $(p_i^{\pm}, \phi_i, \mathbf{A}_{\perp,\,i})$ with $i = 1,2$ are two solutions of the problem (B.1)–(B.3). Since the problem (B.1)–(B.3) has a unique solution (see Theorem 3.1), we have $(p_1^{\pm}, \phi_1, \mathbf{A}_{\perp,\,1}) = (p_2^{\pm}, \phi_2, \mathbf{A}_{\perp,\,2})$ which ends the proof. □

References

1. M. Albrecht-Marc, A. Ghizzo, T. W. Johnston, T. Réveillé, D. Del Sarto and P. Bertrand, Saturation process induced by vortex-merging in numerical Vlasov–Maxwell experiments of stimulated Raman backscattering, *Phys. Plasmas* **14** (2007) 072704.
2. B. B. Afeyan, K. Won, V. Savchenko, T. W. Johnston, A. Ghizzo and P. Bertrand, Kinetic electrostatic electron nonlinear waves and their interactions driven by the ponderomotive force of crossing laser beams, in *Proc. Third Int. Conf. Inertial Fusion Sciences and Applications*, MO34, Monterey, California, 2003, eds. B. Hammel, D. Meyer-Hofer, L. Meyer-ter-Vehn and H. Azechi (American Nuclear Soc., 2004), p. 213.
3. H. L. Berk and K. V. Roberts, *Methods in Computational Physics*, Vol. 9 (Academic Press, 1970).
4. I. B. Bernstein, J. M. Greene and M. D. Kruskal, Exact nonlinear plasma oscillations, *Phys. Rev.* **108** (1957) 546–550.
5. F. Berthelin and F. Bouchut, Solution with finite energy to a BGK system relaxing to isentropic gas dynamics, *Ann. Fac. Sci. Toulouse* **9** (2000) 605–630.
6. F. Berthelin and F. Bouchut, Kinetic invariant domains and relaxation limit from a BGK model to isentropic gas dynamics, *Asympt. Anal.* **31** (2002) 153–176.
7. F. Berthelin and F. Bouchut, Relaxation to isentropic gas dynamics for a BGK system with single kinetic entropy, *Methods Appl. Anal.* **9** (2002) 313–327.
8. P. Bertrand and M. R. Feix, Nonlinear electron plasma oscillation: The water bag model, *Phys. Lett.* **28A** (1968) 68–69.
9. P. Bertrand and M. R. Feix, Frequency shift of nonlinear electron plasma oscillation, *Phys. Lett.* **29A** (1969) 489–490.

10. N. Besse, P. Bertrand, P. Morel and E. Gravier, Weak turbulence theory and simulation of the gyro-water-bag, *Phys. Rev. E* **77** (2008) 056410.
11. N. Besse, F. Berthelin, Y. Brenier and P. Bertrand, The multi-water-bag equations for collisionless kinetic modeling, *Kinetic Relat. Mod.* **2** (2009) 39–80.
12. N. Besse and P. Bertrand, The gyro-water-bag approach in nonlinear gyrokinetic turbulence, *J. Comput. Phys.* **228** (2009) 3973–3995.
13. N. Besse and P. Bertrand, Quasilinear analysis of the gyro-water-bag model, *Europhys. Lett.* **83** (2008) 25003.
14. N. Besse, G. Latu, A. Ghizzo, E. Sonnendrücker and P. Bertrand, A wavelet-MRA-based adaptive semi-Lagrangian method for the relativistic Vlasov–Maxwell system, *J. Comput. Phys.* **227** (2008) 7889–7916.
15. N. Besse, On the waterbag continuum, *Arch. Rational Mech. Anal.* **119** (2011) 453–491.
16. N. Besse, On the Cauchy problem for the gyro-water-bag model, *Math. Models Methods Appl. Sci.* **21** (2011) 1839–1869.
17. M. Bézard, Régularité L^p précisée des moyennes dans les équations de transport, *Bull. Soc. Math. France* **122** (1994) 29–76.
18. F. Bouchut and B. Perthame, Kruzkov's estimates for scalar conservation laws revisited, *Trans. Amer. Math. Soc.* **350** (1983) 2847–2870.
19. Y. Brenier, Résolution d'équations d'évolution quasilinéaires en dimension N d'espace à l'aide d'équations linéaires en dimension $N+1$, *J. Differential Equations* **50** (1983) 375–390.
20. Y. Brenier, Averaged multivalued solutions for scalar conservation laws, *SIAM J. Numer. Anal.* **21** (1984) 1013–1037.
21. Y. Brenier and L. Corrias, A kinetic formulation for multi-branch entropy solutions of scalar conservation laws, *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **15** (1998) 169–190.
22. H. Brezis, *Analyse Fonctionnelle, Théorie et Applications* (Masson, 1987).
23. K. M. Case, Plasma oscillations, *Ann. Phys.* **7** (1959) 349–364.
24. M. G. Crandall and L. Tartar, Some relations between nonexpansive and order preserving mappings, *Proc. Amer. Math. Soc.* **78** (1980) 385–390.
25. A.-L. Dalibard, Kinetic formulation for heterogeneous scalar conservation laws, *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **23** (2006) 475–498.
26. D. C. DePackh, The water-bag model of a sheet electron beam, *J. Electron. Control* **13** (1962) 417–424.
27. R. L. Dewar and J. Lindl, Nonlinear frequency shift of a plasma wave, *Phys. Fluids* **15** (1972) 820–824.
28. R. J. DiPerna, Measure valued solutions to conservation laws, *Arch. Rational Mech. Anal.* **88** (1985) 223–270.
29. R. J. Diperna, P.-L. Lions and Y. Meyer, L^p regularity of velocity averages, *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **8** (1991) 271–287.
30. J. F. Drake, P. K. Kaw, Y. C. Lee, G. Schmid, C. S. Liu and M. N. Rosenbluth, Parametric instabilities of electromagnetic waves in plasmas, *Phys. Fluids* **17** (1974) 778–885.
31. K. Estabrook, W. L. Kruer and B. F. Lasinski, Heating by Raman backscatter and forward scatter, *Phys. Rev. Lett.* **45** (1980) 1399–1403.
32. M. R. Feix, F. Hohl and L. D. Staton, Nonlinear effects in plasmas, in *Plasmas*, eds. G. Kalman and M. R. Feix (Gordon and Breach, 1969), pp. 3–21.
33. U. Finzi, Accessibility of exact nonlinear states in water-bag model computer experiments, *Plasma Phys.* **14** (1972) 327–338.

34. D. W. Forslund, J. M. Kindel and E. L. Lindman, Plasma simulation studies of stimulated scattering processes in laser-irradiated plasmas, *Phys. Fluids* **18** (1975) 1017–1030.
35. A. Ghizzo, D. DelSarto, T. Réveillé, N. Besse and R. Klein, Self-induced transparency scenario revisited via beat-wave heating induced by Doppler shift in overdense plasma layer, *Phys. Plasmas* **14** (2007) 062702.
36. Y. Giga and T. Miyakawa, A kinetic construction of global solutions of first order quasilinear equations, *Duke Math. J.* **50** (1983) 505–515.
37. R. T. Glassey, *The Cauchy Problem in Kinetic Theory* (SIAM, 1996).
38. E. Godlewski and P.-A. Raviart, *Hyperbolic Systems of Conservation Laws*, Mathématiques et applications, Vol. 3/4 (SMAI, 1991).
39. F. Golse, B. Perthame and R. Sentis, Un résultat de compacité pour les équations de transport et application au calcul de la limite de la limite de la valeur propre principale d'un opérateur de transport, *C. R. Acad. Sc., Série I* **301** (1985) 341–344.
40. F. Golse, P.-L. Lions, B. Perthame and R. Sentis, Regularity of the moments of the solution of a transport equation, *J. Funct. Anal.* **76** (1988) 110–125.
41. L. Gosse, Using K-branch entropy solutions for multivalued geometric optics computations, *J. Comput. Phys.* **180** (2002) 155–182.
42. L. Gosse, S. Jin and X. Li, Two moment systems for computing multiphase semiclassical limits of the Schrödinger equation, *Math. Models Methods Appl. Sci.* **13** (2003) 1689–1723.
43. L. Gosse and P. A. Markowich, Multiphase semiclassical approximation of an electron in a one-dimensional crystalline lattice I. Homogeneous problems, *J. Comput. Phys.* **197** (2004) 387–417.
44. E. Hiob and A. J. Barnard, Vlasov simulation of stimulated Raman scattering in one dimension, *Phys. Fluids* **26** (1983) 3119–3126.
45. J. P. Holloway and J. Dorning, Undamped plasma waves, *Phys. Rev. A* **44** (1991) 3856–3868.
46. S. Jin and X. Li, Multi-phase computations of the semiclassical limit of the Schrödinger equation and related problems: Whitham vs. Wigner, *Physica D* **182** (2003) 46–85.
47. S. N. Kruzkov, Generalized solutions of the Cauchy problem in the large for nonlinear equations of first order, *Sov. Math. Dokl.* **10** (1969) 785–788 .
48. S. N. Kruzkov, First order quasilinear equations in several independent variables, *Math. USSR Sb.* **10** (1970) 217–243 .
49. C. Lancelloti and J. J. Dorning, Time-asymptotic traveling-wave solutions to the nonlinear Vlasov–Poisson–Ampère equations, *J. Math. Phys.* **40** (1999) 3895–3917.
50. C. Lancelloti and J. J. Dorning, Time-asymptotic wave propagation in collisionless plasmas, *Phys. Rev. E* **68** (2003) 026406.
51. L. D. Landau, On vibrations of electronic plasma, *Zh. Eksp. Teor. Fiz.* **16** (1946) 574–586.
52. J. L. Lions and E. Magenes, *Problèmes aux Limites Non homogènes* (Dunod, 1968).
53. P.-L. Lions and B. Perthame, Propagation of moments and regularity for the three-dimensional Vlasov–Poisson system, *Invent. Math.* **105** (1991) 415–430.
54. P.-L. Lions, B. Perthame and E. Tadmor, A kinetic formulation of multidimensional scalar conservation laws and related equations, *J. Amer. Math. Soc.* **7** (1994) 169–191.
55. P.-L. Lions, B. Perthame and E. Tadmor, Kinetic formulation of isentropic gas dynamics and p-systems, *Commun. Math. Phys.* **163** (1994) 415–431.
56. P. Morel, E. Gravier, N. Besse, A. Ghizzo and P. Bertrand, The water bag and gyrokinetic applications, *Comm. Nonlinear Sci. Numer. Simulat.* **13** (2008) 11–17.

57. P. Morel, E. Gravier, N. Besse, R. Klein, A. Ghizzo, P. Bertrand, X. Garbet, P. Ghendrih, V. Grandgirard and Y. Sarazin, Gyro kinetic modelling: A multi-water-bag approach, *Phys. Plasmas* **14** (2007) 112109.
58. C. Mouhot and C. Villani, Landau damping, *J. Math. Phys.* **51** (2010) 015204, http://www.umpa.ens-lyon.fr/c̃villani/ or arXiv:0904.2760.
59. B. Perthame, *Kinetic Formulation of Conservation Laws*, Oxford Lectures Series in Mathematics and its Applications, Vol. 21 (Oxford Univ. Press, 2002).
60. B. Perthame, Uniqueness and error estimates in first order quasilinear conservation laws via the kinetic entropy defect measure, *J. Math. Pures Appl.* **77** (1998) 1055–1064.
61. B. Perthame and E. Tadmor, A kinetic equation with kinetic entropy functions for scalar conservation laws, *Commun. Math. Phys.* **136** (1991) 501–517.
62. A. Szepessy, An existence result for scalar conservation laws using measure valued solutions, *Commun. Partial Differential Equations* **14** (1989) 1329–1349.
63. N. G. Van Kampen, On the theory of stationary waves in plasmas, *Physica* **21** (1955) 949–963.
64. A. Vasseur, Kinetic semidiscretization of scalar conservation laws and convergence by using averaging lemmas, *SIAM J. Numer. Anal.* **36** (1999) 465–474.
65. A. Vasseur, Convergence of a semi-discrete kinetic scheme for the system of isentropic gas dynamics with $\gamma = 3$, *Univ. Math. J.* **48** (1999) 347–364.
66. A. I. Volpert, The space BV and quasilinear equations, *Math. USSR Sb.* **2** (1967) 225–267.