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ON THE CAUCHY PROBLEM FOR THE GYRO-WATER-BAG MODEL

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In this paper we prove the existence and uniqueness of classical solution for a system of PDEs recently developed in Refs. 60, 8, 10 and 11 to modelize the nonlinear gyrokinetic turbulence in magnetized plasma. From the analytical and numerical point of view this model is very promising because it allows to recover kinetic features (wave−particle interaction, Landau resonance) of the dynamic flow with the complexity of a multi-fluid model. This model, called the gyro-water-bag model, is derived from two-phase space variable reductions of the Vlasov equation through the existence of two underlying invariants. The first one, the magnetic moment, is adiabatic and the second, a geometric invariant named "water-bag", is exact and is just the direct consequence of the Liouville theorem.

Keywords: Gyro-water-bag model; collisionless kinetic equations; Cauchy problem; hyperbolic systems of conservation laws; gyrokinetic turbulence; Vlasov equation; plasma physics; pseudo-differential operators.

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1. Introduction

It is generally recognized that the anomalous transport observed in nonuniform magnetized plasmas is related to the existence of turbulent low-frequency electromagnetic fluctuations, i.e. with frequency much lower than the ion gyrofrequency.

The presence of density, temperature and velocity gradients in the transverse direction of the magnetic confinement field, generates micro-instabilities which give rise to this turbulent transport. Low frequency ion-temperature-gradient-driven instability is one of the most serious candidates to account for the anomalous transport,[74] as well as the so-called trapped electron modes.[62] As the main energy loss in a controlled fusion devices is of conductive nature, the energy confinement time is of the same order as the diffusion time a^2/χ_T where χ_T is the thermal diffusivity and "a" the transverse plasma size. Therefore it is crucial to determine this transport coefficient by computing the turbulent nonlinear diffusivities in fusion plasmas. During recent years, ion turbulence in tokamaks has been intensively studied both with fluid[26,37,56] and gyrokinetic simulations using Particle-In-Cell (PIC) codes[46,47,52,53,63,68] or Vlasov codes.[20,23,27,43]

As far as the turbulent diffusion is concerned, it is commonly observed[24] that there exists a factor 2 between kinetic and fluid simulations ($\chi_{\text{fluid}} > 2\chi_{\text{kinetic}}$). Therefore the kinetic or fluid description may significantly impact the instability threshold as well as the predicted turbulent transport. The reasons of this observation is not really well understood: nonlinear Landau effects or nonlinear resonant wave−particle interaction, damping of poloidal velocity fluctuations, and so on.

Consequently, it is important that gyrokinetic simulations measure the discrepancy between the local distribution function and a Maxwellian one, which is the main assumption of fluid closures.

In a recent paper Ref. 67 a comparison between fluid and kinetic approach has been addressed by studying a three-dimensional kinetic interchange. A simple driftkinetic model is described by a distribution depending only on two spatial dimensions and parametrized by the energy. In that case it appears that the distribution function is far from a Maxwellian and cannot be described by a small number of moments. Wave−particle resonant processes certainly play an important role and most of the closures that have been developed will be inefficient.

On the other hand, although more accurate, the kinetic description of turbulent transport is much more demanding in computer resources than fluid simulations. This motivated us to revisit an alternative approach based on the water-bag-like weak solution of Vlasov-gyrokinetic equations.

The water-bag model was introduced initially by DePackh,[22] Hohl, Feix and Bertrand,[5,6,30] next extended to a double water-bag by Berk and Roberts[2] and finally generalized to the multiple water-bag by Finzi.[3,4,7,31,61] The water-bag model was shown to bring the bridge between fluid and kinetic description of a collisionless plasma, allowing to keep the kinetic aspect of the problem (wave−particle interaction, Landau resonance) with the same complexity as a multi-fluid model. The aim of this paper is to use the water-bag description for gyrokinetic modeling. In order to understand the nature of the transport, the weak-turbulence theory of the gyro-water-bag has been developed in Ref. 8 leading to nonlinear Fokker−Planck equations for the bags (revealing the diffusive nature of the transport in the radial direction) coupled to a diffusion equation for the mean flow (zonal flow) which constitutes the back reaction (inverse energy cascade) of turbulent diffusion (direct energy cascade). Indeed there is an energy transfer from the turbulent low-frequency electromagnetic (drift waves) fluctuations to these periodic zonal flow fluctuations via either local or nonlocal interactions in Fourier space. The back reaction of self-generated shear flow (such as both radially sheared parallel and poloidal flows), on pressure-gradient-driven turbulence, is a key mechanism that governs the turbulent state and the transport, especially it can lead to the formation of transport barriers which participates to a better confinement of the plasma. Through a quasilinear analysis it has been derived semi-analytical transport coefficients to predict the level of the turbulence. In Ref. 10 another quasilinear model, well-suited for numerical simulation of weak turbulence of magnetized plasma in a cylinder, is derived. This quasilinear model is solved using a numerical approximation scheme based on discontinuous Galerkin methods. Finally the full nonlinear gyro-water-bag model is solved numerically in Ref. 9 by the means of Runge–Kutta semi-Lagrangian methods. The comparison of numerical results between nonlinear and quasilinear simulations[10] show that the quasilinear approach proves to be a good approximation of the full nonlinear one as the quasilinear estimate of the turbulent transport is of the same order as the nonlinear one. In order to show the relevance of the gyro-water-bag model for describing plasma nonlinear gyrokinetic turbulence, we are now making numerical comparisons between the nonlinear gyro-water-bag model (for which we will prove well-posedness below) and the Vlasov-gyrokinetic equations thanks to the GYSELA[43] code developed at the CEA-Cadarache. After a brief introduction of the well-known gyrokinetic equations hierarchy, we present the derivation of the gyro-water-bag model. We next show the existence and uniqueness of classical solution of the gyro-water-bag model.

2. The Gyro-Water-Bag Model

2.1. *The gyrokinetic equation*

Predicting turbulent transport in collisionless fusion plasmas requires to solve the gyrokinetic equations for all species coupled to Darwin or Poisswell equations (low-frequency approximations of Maxwell equations in the asymptotic limit of infinite speed of light[11]). This gyrokinetic approach has been widely used in recent years to study low-frequency micro-instabilities in magnetically confined plasmas which are known for exhibiting a wide range of spatial and temporal scales. Gyrokinetic ordering employs the fact that the characteristic frequencies of the waves and gyroradii are small compared with the gyrofrequencies and unperturbed scale lengths, respectively, and that the perturbed parallel scale lengths are of the order of the unperturbated scale length. Such an ordering enables one to be rid of the explicit dependence on the phase angle of the Vlasov equation through gyrophase-averaging while retaining the gyroradius effects to the arbitrary values of the gyroradius over the perturbated perpendicular scale length. The conventional approach[36] to derive the gyrokinetic Vlasov equation is based on a maximal multiple-scale-ordering expansion involving a single ordering parameter, which consists in computing an iterative solution of the gyroangle-averaged Vlasov equation perturbatively expanded in powers of a dimensionless parameter ρ/L, where ρ is the Larmor radius and L, the characteristic background magnetic-field or plasma density and temperature nonuniformity length scale. A modern foundation of nonlinear gyrokinetic theory[19,28,44] is based on a two-step Lie-transform approach. The first step consists in the derivation of the guiding-center Hamilton equations, from the Hamiltonian particle dynamics, through the elimination of the gyroangle associated with the gyromotion time-scale of charged particles. If one takes into account finite gyroradius effects, one needs to reintroduce the gyroangle dependence into the perturbated guiding-center Hamiltonian dynamics which results that the magnetic moment μ is only conserved at first order in the dimensionless ordering parameter featuring electrostatic perturbations. Therefore one needs to perform a second-order perturbation analysis to derive the nonlinear gyrocenter dynamics. As a result, the second step consists in deriving a new set of gyrocenter Hamiltonian equations from the perturbated guiding-center equations, through a time-dependent gyrocenter phase-space transformation and gyroangle elimination. Finally, a reduced variational principle[18,19] enables to derive self-consistent expressions for the nonlinear gyrokinetic Vlasov Maxwell equations. Within gyrokinetic Hamiltonian formalism, the Vlasov equation expresses the fact that the ions gyrocenter distribution function $f = f(t, \mathbf{r}, v_\parallel, \mu)$ is constant along gyrocenter characteristic curves in gyrocenter phase-space $(t, \mathbf{r}, v_\parallel, \mu)$:

$$D_t f = \partial_t f + \dot{X}_\perp \cdot \nabla_\perp f + \dot{X}_\parallel \cdot \nabla_\parallel f + \dot{v}_\parallel \partial_{v_\parallel} f = 0, \tag{2.1}$$

with

$$\dot{X}_\parallel = v_\parallel \mathbf{b}, \quad \dot{X}_\perp = \mathbf{v_E} + \mathbf{v}_{\nabla B} + \mathbf{v}_c,$$

$$\mathbf{v_E} = \frac{1}{B_\parallel^*}\mathbf{b} \times \nabla \mathcal{J}_\mu \phi,$$

$$\mathbf{v}_{\nabla B} = \frac{\mu}{q_i B_\parallel^*}\mathbf{b} \times \nabla B,$$

$$\mathbf{v}_c = \frac{m_i v_\parallel^2}{q_i B_\parallel^*}\left(\frac{\mathbf{b} \times \nabla B}{B} + \frac{(\nabla \times \mathbf{B})_\perp}{B}\right) = \frac{m_i v_\parallel^2}{q_i B_\parallel^*}\mathbf{b} \times \frac{\mathbf{N}}{R_c},$$

$$\dot{v}_\parallel = -\frac{1}{m_i}\left(\mathbf{b} + \frac{m_i v_\parallel}{q_i B_\parallel^*}\mathbf{b} \times \frac{\mathbf{N}}{R_c}\right) \cdot (\mu \nabla B + q_i \nabla \mathcal{J}_\mu \phi),$$

$$\mathbf{B}^* = \mathbf{B} + \frac{m_i v_\parallel}{q_i} \nabla \times \mathbf{b}, \quad B_\parallel^* = \mathbf{B}^* \cdot \mathbf{b},$$

where $\mathbf{b} = \mathbf{B}/B$ denotes the unit vector along magnetic field line, $\mathcal{J}_\mu$ denotes the gyroaverage operator, $\mathbf{N}/R_c$ is the field line curvature, $q_i = Z_i e$, $e > 0$ being the electron Coulomb charge and $\mu = m_i v_\perp^2/(2B)$ is the first adiabatic invariant of the ion gyrocenter. If we now suppose $k_\perp \rho_i$ small and neglecting all terms smaller than $\mathcal{O}(k_\perp^2 \rho_i^2)$, we then obtain the Poisson equation

$$-Z_i q_i \nabla_\perp \cdot \left(\frac{n_i \rho_i^2}{k_\mathrm{B} T_i} \nabla_\perp \phi \right) = \lambda_{D_i}^2 Z_i q_i \frac{n_i}{k_\mathrm{B} T_i} \Delta\phi + Z_i \int 2\pi \frac{\Omega_i}{q_i} d\mu dv_\parallel \mathcal{J}_\mu f - n_e, \tag{2.2}$$

where $\rho_i^2 = v_{thi}^2/\Omega_i^2 = k_\mathrm{B} T_i/(m_i \Omega_i^2)$ is the ion Larmor radius, $\lambda_{D_i}^2 = k_\mathrm{B} T_i \varepsilon_0/(Z_i^2 e^2 n_i)$ is the ion Debye length and J_0 is the Bessel function of zero order. The left-hand side of Eq. (2.2) corresponds to the difference between the gyroaveraged density $\frac{\Omega_i}{q_i} \times \int d\mu dv_\parallel \mathcal{J}_\mu f$ and the laboratory ion density N_i which is the lowest contribution to the density fluctuations provided by the polarization drift. Firstly, we are interested in the effects of the transversal drift velocity $\mathbf{E} \times \mathbf{B}$ coupled to the parallel dynamics while the curvature effects are considered as a next stage of the study. As a result, in the sequel we deal with a reduced driftkinetic model in cylindrical geometry by making the following approximations.

- In addition of cylindrical geometry, we suppose that the magnetic field $\mathbf{B}$ is uniform and constant along the axis of the column (z-coordinate, $\mathbf{B} = B\mathbf{b} = B\mathbf{e}_z$). It follows the perpendicular drift velocity does not admit any magnetic curvature or gradient effect and especially $\mathbf{B}^* = \mathbf{B}$.
- Some finite Larmor radius effects are neglected. Namely we consider only one adiabatic invariant $\mu_i = \rho_i^2 \Omega_i q_i/2$ and set $\mathcal{J}_{\mu_i} = 1$ which means that the asymptotic $k_\perp \rho_i \to 0$ is considered and thus the guiding center and the gyrocenter merge.
- We linearize the expression for the polarization density, n_pol, in Eq. (2.2),

$$n_\mathrm{pol} = \nabla_\perp \cdot \left(\frac{n_i}{B\Omega_{ci}} \nabla_\perp \phi \right),$$

  by approximating n_i to the background density of the Maxwellian distribution function n_{i0}, and by assuming that the ion cyclotron frequency, Ω_i is a constant Ω_0. Moreover we assume that the ion Debye length λ_{D_i} is small compared to the ion Larmor radius ρ_i.
- The electron inertia is ignored, i.e. we choose an adiabatic response to the low frequency fluctuations for the electrons. In other words the electron density follows the Boltzmann distribution

$$n_e = n_{e0} \exp\left(\frac{e}{k_\mathrm{B} T_e} (\phi - \lambda \langle \phi \rangle_\mathcal{M}) \right),$$

  where $\langle \phi \rangle_\mathcal{M}$ denotes the average of the electrical potential ϕ over a magnetic field line. Moreover we assume that the electrical potential is small compared to the electron kinetic energy $e\phi/(k_\mathrm{B} T_e) \sim \varepsilon_\delta \ll 1$.

Under these assumptions the evolution of the ion guiding-center distribution function $f = f(t, \mathbf{r}_\perp, z, v_\parallel)$ obeys the driftkinetic Vlasov equation

$$\partial_t f + \mathbf{v_E} \cdot \nabla_\perp f + v_\parallel \partial_z f + \frac{q_i}{m_i} E_\parallel \partial_{v_\parallel} f = 0, \tag{2.3}$$

for the ions (q_i, M_i), coupled to an adiabatic electron response via the quasi-neutrality assumption

$$-\nabla_\perp \cdot \left(\frac{n_{i0}}{B\Omega_0} \nabla_\perp \phi \right) + \frac{e\tau n_{i0}}{T_{i0}} (\phi - \lambda \langle \phi \rangle_{\mathcal{M}}) = \int_{\mathbb{R}} f(t, \mathbf{r}, v_\parallel) dv_\parallel - n_{i0}. \tag{2.4}$$

Here $q_i = Z_i e$, $Z_i n_{i0} = n_{e0}$, $T_e = T_{e0}$, $\tau = T_{i0}/T_{e0}$, $\lambda \in \{0, 1\}$, $\mathbf{E} = -\nabla\phi$ and $\mathbf{v_E}$ is the $\mathbf{E} \times \mathbf{B}/B^2$ drift velocity. The most important and interesting feature is that f depends, through a differential operator, only on the velocity component $v_\parallel$ parallel to $\mathbf{B}$. Let us note that rigorous mathematical justifications of Vlasov-gyrokinetic models (with the full three-dimensional Poisson equation) in simplified geometry (with no magnetic curvature neither magnetic gradient drift velocity) with various time and space scales ordering have been performed recently in different configurations ("transverse guiding center approximation", "parallel approximation", "finite Larmor radius approximation", "quasi-neutral limit").[16,32–35,41,42,65,66]

2.2. *The gyro-water-bag model*

Let us now turn back to the driftkinetic equation (2.3). Since the distribution $f(t, \mathbf{r}_\perp, z, v_\parallel)$ takes into account only one velocity component $v_\parallel$ a water-bag solution can be considered.[3] Let us consider $2\mathcal{N}$ non-closed contours in the $(\mathbf{r}, v_\parallel)$-phase space labeled v_j^+ and v_j^- (where $j = 1, \ldots, \mathcal{N}$) such that $\cdots < v_{j+1}^- < v_j^- < \cdots < 0 < \cdots < v_j^+ < v_{j+1}^+ < \cdots$ and some strictly positive real numbers $\{\mathcal{A}_j\}_{j \in [1, \mathcal{N}]}$ that we call bag heights. Since the bags $\mathcal{A}_j(v_j^+ - v_j^-)$, for $j = 1, \ldots, \mathcal{N}$, are exact geometric invariants, which are reminiscent to the geometric Liouville invariant, we then define $f_{\mathrm{gwb}} = f(t, \mathbf{r}_\perp, z, v_\parallel)$ as

$$f(t, \mathbf{r}_\perp, z, v_\parallel) = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_j [\mathcal{H}(v_\parallel - v_j^-(t, \mathbf{r}_\perp, z)) - \mathcal{H}(v_\parallel - v_j^+(t, \mathbf{r}_\perp, z))], \tag{2.5}$$

where $\mathcal{H}$ is the Heaviside unit step function. The function (2.5) is an exact solution of the driftkinetic Vlasov equation (2.3) in the sense of distribution theory if and only if the set of following equations are satisfied:

$$\partial_t v_j^\pm + \mathbf{v_E} \cdot \nabla_\perp v_j^\pm + v_j^\pm \partial_z v_j^\pm = \frac{q_i}{m_i} E_\parallel. \tag{2.6}$$

The quasi-neutrality coupling can be rewritten as

$$-\nabla_\perp \cdot \left(\frac{n_{i0}}{B\Omega_0} \nabla_\perp \phi \right) + \frac{e\tau n_{i0}}{T_{i0}} (\phi - \lambda \langle \phi \rangle_{\mathcal{M}}) = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_j (v_j^+ - v_j^-) - n_{i0}. \tag{2.7}$$

Let us introduce for each bag j the density $n_j = (v_j^+ - v_j^-)\mathcal{A}_j$ and the average velocity $u_j = (v_j^+ + v_j^-)/2$. After a little algebra, Eq. (2.6) leads to continuity and Euler equations namely

$$\partial_t n_j + \nabla_\perp \cdot (n_j \mathbf{v_E}) + \partial_z (n_j u_j) = 0, \tag{2.8}$$

$$\partial_t (n_j u_j) + \nabla_\perp \cdot (n_j u_j \mathbf{v_E}) + \partial_z \left(n_j u_j^2 + \frac{p_j}{m_i} \right) = \frac{q_i}{m_i} n_j E_\parallel, \tag{2.9}$$

where the partial pressure takes the form $p_j = m_i n_j^3/(12\mathcal{A}_j^2)$. The connection between kinetic and fluid description clearly appears in the previous multi-fluid equations. The case of one bag recovers a fluid description (with an exact adiabatic closure with $\gamma = 3$) and the limit of an infinite number of bags provides a continuous distribution function.

To complete the system (2.6)–(2.7) we need to supply an initial condition $v_j^\pm(t = 0, \mathbf{r}_\perp, z) = v_{0j}^\pm(\mathbf{r}_\perp, z)$ for $j \in [1, \mathcal{N}]$. In fact, the problem of determinating the water-bag parameters $(\{\mathcal{A}_j\}_{j\in[1,\mathcal{N}]}, \{v_{0j}^\pm\}_{j\in[1,\mathcal{N}]})$ is not a trivial task. From a general framework point of view, we can minimize a distance, which has to be suitably defined, between any given distribution function belonging to some functional spaces and the water-bag distribution function (2.5) under some appropriate constraints (for example, on the sign of the parameters $\{\mathcal{A}_j\}_{j\in[1,\mathcal{N}]}$ which must lead to the definition of a positive measure in velocity space). For example, we can decide to minimize the distance between the moments of any given distribution function and the water-bag decomposition (2.5). This kind of moment problem under constraints can be recast in a general nonlinear optimization problem with constraints. As an example, to determine physically relevant gyro-water-bag equilibrium to describe ion-temperature-gradient modes, we can choose to construct radial profiles in terms of temperature and density profiles only. The continuous equilibrium distribution function can be assumed as

$$f_{\mathrm{eq}}(r, v_\parallel) = \frac{n_{i0}(r)}{\sqrt{T_{i0}(r)}} \mathcal{F}\left(\frac{v_\parallel}{\sqrt{T_{i0}(r)}} \right), \tag{2.10}$$

where $n_{i0}(r)$ and $T_{i0}(r)$ are normalized radial profiles of ion density and temperature, and $r = |\mathbf{r}_\perp|$. We can suppose that the normalized function $\mathcal{F}$ is an even function, which leads to consider symmetric profiles such that $v_{0j}^\pm = \pm v_{0j}(r_0)$. As an example for a local Maxwellian distribution, we get $\mathcal{F}(x) = \exp(-x^2/2)/\sqrt{2\pi}$. The first stage can then consist in constructing the gyro-water-bag equilibrium function at $r = r_0 \in [r_{\min}, r_{\max}]$, while a second one can consist in extending it on the whole radial domain $[r_{\min}, r_{\max}]$. To this aim, as in Refs. 60, 9, 21, we can use the method of equivalence between the moments of the stepwise gyro-water-bag function (2.5) and the corresponding continuous function (2.10). If for a given set of contours $\{v_{0j}(r_0)\}_{j\in[1,\mathcal{N}]}$, with $\ell = 0, 2, \ldots, 2(\mathcal{N} - 1)$, we equalize $\mathcal{M}^\ell(f_{\mathrm{eq}})$, the ℓ-moment in velocity of f_{eq}, and $\mathcal{M}^\ell(f_{\mathrm{gwb}}^0)$, the ℓ-moment in velocity of the gyro-water-bag function $f_{\mathrm{gwb}}^0 = f_{\mathrm{gwb}}(t = 0)$ (see definition (2.5)), at $r = r_0$, we obtain a $\mathcal{N}$-dimensional linear problem where the unknowns are the parameters $\{\mathcal{A}_j\}_{j\in[1,\mathcal{N}]}$ and the matrix $\mathcal{L}^\dagger$ of the linear problem is the Vandermonde-type matrix $\{\mathcal{L}_{ij}^\dagger = 2v_{0j}^{2i-1}(r_0)\}_{i,j\in[1,\mathcal{N}]}$, while the right-hand side involves the moments $\mathcal{M}^\ell(f_{\rm eq})$. At this point there are several strategies to extend the contours $\{v_{0j}\}_{j\in[1,\mathcal{N}]}$ on the whole radial domain. One strategy is to follow the level lines of the distribution function $\mathcal{F}$, determinated by the method of moments equivalence at $r = r_0$, which leads to the definition $v_{0j}(r) = \sqrt{T_{i0}(r)}\mathcal{F}^{-1}(f_j\sqrt{T_{i0}(r)}/n_{i0}(r))$, with $f_j = f_{\rm eq}\big(r_0, v_{0j}(r_0)\big)$. If $n_{i0}(r)$, $T_{i0}(r)$ and $\mathcal{F}$ are enough regular, then it will be also the case for the initial contours $\{v_{0j}\}_{j\in[1,\mathcal{N}]}$. A second strategy can consist in differentiating with respect to the radial variable r the moments equivalence $\mathcal{M}^\ell(f^0_{\rm gwb}) = \mathcal{M}^\ell(f_{\rm eq})$ at $r = r_0$, which leads to a $\mathcal{N}$-dimensional linear problem where the unknowns are now the radial derivatives of the contours $\{\partial_r v_{0j}(r_0)\}_{j\in[1,\mathcal{N}]}$ and the matrix of the linear problem is the Vandermonde-type matrix $\{\mathcal{L}^\ddagger_{ij} = 2\mathcal{A}_j v_{0j}^{2(i-1)}(r_0)\}_{i,j\in[1,\mathcal{N}]}$, while the right-hand side involves now the known quantities $(\partial_r\mathcal{M}^\ell(f_{\rm eq}))(r_0)$, and $\{v_{0j}(r_0)\}_{j\in[1,\mathcal{N}]}$. By differentiating the moments equivalence a second time with respect to radius r, we still obtain a $\mathcal{N}$-dimensional linear problem of matrix $\mathcal{L}^\ddagger$, where the unknowns are now the second-order radial derivatives of the contours $\{\partial^2_r v_{0j}(r_0)\}_{j\in[1,\mathcal{N}]}$, while the right-hand side involves now the known quantities $(\partial^k_r\mathcal{M}^\ell(f_{\rm eq}))(r_0)$ and $\{\partial^l_r v_{0j}(r_0)\}_{j\in[1,\mathcal{N}]}$ with $k \le 2$ and $l \le 1$. Following the same previous procedure we can obtain any high-order radial derivatives of the contours $\{\partial^m_r v_{0j}(r_0)\}_{j\in[1,\mathcal{N}]}$ by solving $\mathcal{N}$-dimensional linear problems of matrix $\mathcal{L}^\ddagger$, where the right-hand side involves the radial derivatives $(\partial^k_r\mathcal{M}^\ell(f_{\rm eq}))(r_0)$ and $\{\partial^l_r v_{0j}(r_0)\}_{j\in[1,\mathcal{N}]}$ with $k \le m$ and $l \le m-1$. Using the mth first radial derivatives of the contours $\{v_{0j}(r_0)\}_{j\in[1,\mathcal{N}]}$ at $r = r_0$ we can extrapolate the values of $\{v_{0j}(r_0+\delta r)\}_{j\in[1,\mathcal{N}]}$ at $r_0 + \delta r$ by using a Taylor expansion. Finally, we can repeat the whole previous process at the point $r = r_0 + \delta r$. Knowing the values of the contours and their radial derivatives at any order on a grid of the radial domain, i.e. $\{\partial^k_r v_{0j}(r_i)\}_{j\in[1,\mathcal{N}],i\in[1,M]}$, with $k \le m$, we can use an interpolation scheme of high regularity (such that Hermite or B-splines interpolation) to construct initial contours with the desired regularity.

Let us notice that after a finite time, Eq. (2.6) or the system (2.8)–(2.9) will generate shocks, namely discontinuous gradients in z for $v_j^\pm$, n_j and u_j. Nevertheless the concept of entropic solution is not well-suited here because the existence of an entropy inequality means that a diffusion-like (or scattering-like) process in velocity occurs on the right-hand side of the Vlasov equation. This observation has been developed in the theory of kinetic formulation of scalar conservation laws. In fact it was established in Refs. 13–15 and 38 that scalar conservation laws can be lifted as linear hyperbolic equations by introducing an extra variable $\xi \in \mathbb{R}$ which can be interpreted as a scalar momentum or velocity variable. The author of Ref. 15 proposed a numerical scheme, known as the transport-collapse method to solve this linear kinetic equation. In fact the solution of this numerical scheme can be seen as the solution of a variant version of the linear Bhatnagar–Gross–Krook (BGK) kinetic model. The authors of Refs. 14, 15 and 38 have proved, using BV estimates and Kruzhkov-type analysis, that this numerical solution converges to the entropy solution of scalar conservation laws. This result was also shown in Ref. 72 using averaging lemmas[39,40,25,12] without bounded variation (BV) estimates. In Ref. 64 the authors also consider the BGK-like approximation, and using again BV estimates, they prove the convergence of the approximate solution to the right entropy solution when the relaxation time (the inverse of the collisional frequency) tends to zero. Right after, it was observed by the authors of Refs. 64 and 54 that, without any approximations, entropy solutions of scalar conservation laws can be directly formulated in kinetic style, known as kinetic formulation. Its generalization to systems of conservation laws seems impossible except for very peculiar systems.[17,55,73] Actually velocity derivatives of non-negative bounded measure appear in the right-hand side of these linear kinetic equations (free streaming terms), which is the signature of diffusion-like processes in velocity. In order that the water-bag model should be equivalent to the Vlasov equation (without any diffusion-like term on the right-hand side of the Vlasov equation) we must consider multivalued solution of the water-bag model beyond the first singularity. The appearance of a singularity (discontinuous gradients in z due to the Burgers term) is linked to appearance of trapped particles which is characterized by the formation of vortices and the development of the filamentation process in the phase space. From the study of particles dynamic,[45] in a cylinder (the geometry for which the gyro-water-bag equations (2.6)−(2.7) are valid) the particles are not trapped but only passing. However, this model is relevant for studying gyrokinetic turbulence in magnetically confined thermonuclear fusion plasmas, because, in cylindrical geometry, wave breaking or filamentation process are not dominant mechanisms.

To the best of my knowledge, until now there is no analytical result concerning the well-posedness of the Vlasov-gyrokinetic equations (2.3)−(2.4) because it is a hard problem to deal with the strong coupling $\phi = n$ along the parallel direction (loss of z-derivatives). It is still an open problem to prove the existence of classical and weak solutions (even locally in time) for the system (2.3)−(2.4). Concerning weak solutions, it seems that traditional techniques, for getting compactness of sequences of approximated solutions, such as averaging lemmas or compensated compactness tools, fail. Maybe the use of relative entropy method would allow to pass to the limit. Therefore the present analytical result constitutes a first step to prove the existence of weak solutions (at least for a special class) for the Vlasov-gyrokinetic equations (2.3)−(2.4). Let us notice that from the physical point of view, any Lebesgue integrable distribution function f, having a finite number of bounded moments, can be approximated by a water-bag distribution function by equating their moments up to a fixed order.[9,21,59] In order to recover some regularity in the parallel direction and then prove the existence of global weak solutions an interesting idea might be to add a diffusion (collision) term in the direction of parallel velocity on the right-hand side of the Vlasov equation (2.3) such as Fokker−Planck-like collision operators. Another way could be to consider a non-Boltzmannian electrons distribution function. In this case the Debye length are comparable to the electrons Larmor radius so that we cannot neglect the Laplacian operator in Eq. (2.2).

3. Existence of Classical Solution for the Gyro-Water-Bag Model

In this section we want to study the existence and uniqueness of the system (2.6)–(2.7). In general the density n_{i0} and the temperature T_{i0} appearing in Eq. (2.7) are smooth given functions of the radius r. To simplify the proof and without loss of generality we can suppose that the density n_{i0} and the temperature T_{i0} are uniform and take $\lambda = 0$ in Eq. (2.7). Therefore the dimensionless equations (2.6)–(2.7) read in $\mathbb{R}^3$ as follows:

$$\partial_t v_j^\pm - \nabla_\perp^\perp \phi \cdot \nabla_\perp v_j^\pm + v_j^\pm \partial_z v_j^\pm + \partial_z \phi = 0, \quad v_j^\pm(0,\cdot) = v_{0j}^\pm(\cdot), \quad j = 1,\dots,\mathcal{N}, \tag{3.1}$$

$$-\Delta_\perp \phi + \phi = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_j (v_j^+ - v_j^-) - 1. \tag{3.2}$$

In the transverse $\mathbf{r}_\perp$-direction the contours follow the dynamics of the inviscid incompressible Euler equations written in vorticity formulation. In the longitudinal z-direction the contours follow the dynamics of Burgers-type equations, where the flux functions involve a nonlocal term only in the transverse direction which couples all the equations. The loss of derivatives is in the z-direction while the gain is in the $\mathbf{r}_\perp$-direction, which makes the problem quite challenging. In order to prove the existence and uniqueness of the gyro-water-bag system (3.1)–(3.2) we split the global dynamic system into the transverse dynamic system and the longitudinal one. For each system we then prove the existence and uniqueness of classical solutions and get *a priori* estimates on this solution. The idea of the proof then consists to construct an approximate solution sequence for the global dynamic system and, thanks to *a priori* estimates on the transverse and longitudinal systems, show that there exists a unique limit which satisfies the exact global dynamic system. The main difficulty of the proof comes from the loss of z-derivatives on the electrical potential ϕ in Eq. (3.2) which leads to a loss of regularity in the z-direction. To overcome this difficulty the trick is to recast the longitudinal dynamic equations into a hyperbolic system of conservation laws.

3.1. *The transverse dynamic system*

In this section, we consider the initial value problem in $\mathbb{R}^3$,

$$\begin{aligned} &\partial_t v_j^\pm - \nabla_\perp^\perp \phi \cdot \nabla_\perp v_j^\pm = 0, \quad v_j^\pm(0,\cdot) = v_{0j}^\pm(\cdot), \quad j = 1,\dots,\mathcal{N}, \\ &-\Delta_\perp \phi + \phi = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_j (v_j^+ - v_j^-) - 1. \end{aligned} \tag{3.3}$$

Therefore we have the following existence theorem.

Theorem 3.1. (Local classical solution) *Assume $v_{0j}^\pm \in H^s(\mathbb{R}^3)$ with $s > n/2 + 1$, $n = 3$. Then for all $\mathcal{N}$ there exists a time $T > 0$ that depends only on $\|v_{0j}^\pm\|_{H^s}$, $\mathcal{N}$ and $A = \max_{j \le \mathcal{N}} |\mathcal{A}_j|$, such that Eq. (3.3) have a unique solution*

$$v_j^\pm \in L^\infty(0,T; H^s(\mathbb{R}^3)) \cap \mathrm{Lip}(0,T; H^{s-1}(\mathbb{R}^3)), \quad j = 1,\dots,\mathcal{N}.$$

Proof. The proof is based on the Banach's fixed-point theorem. We first rewrite the system (3.3). Using the Green function $G(x_\perp, y_\perp) = K_0(|x_\perp - y_\perp|)/(2\pi)$ where K_0 is the modified Bessel function of the second kind of order zero, i.e. the fundamental solution of the differential operator $(1 - \Delta)$ in $\mathbb{R}^2$, we can reconsider the problem (3.3) as

$$\partial_t v_j^\pm + u[\{v_j^\pm\}_{j\le\mathcal{N}}] \cdot \nabla_\perp v_j^\pm = 0,$$

where

$$u[\{v_j^\pm\}_{j\le\mathcal{N}}](t,x) = (\mathbb{K} \overset{\perp}{*} \mathcal{A}_\# \cdot V)(t,x) = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_j \mathbb{K} \overset{\perp}{*} (v_j^+(t,x) - v_j^-(t,x)). \tag{3.4}$$

In expression (3.4) we have used the notations of Sec. 3.2 and we define $\mathbb{K}(x_\perp, y_\perp) = \mathbb{K}(|x_\perp - y_\perp|) = -\nabla_\perp^\perp G(x_\perp, y_\perp) = K_1(|x_\perp - y_\perp|)(x_\perp - y_\perp)^\perp/(2\pi|x_\perp - y_\perp|)$, where K_1 is the modified Bessel function of second kind of order one. Let us note that $\mathbb{K}_i(|\cdot|) \in L^1(\mathbb{R}^2)$ for $i = 1, 2$. We now define the set W_T as

$$\begin{aligned} W_T := \Bigg\{ & w_j^\pm \in L^\infty(0,T; H^s(R^3)) \cap \mathrm{Lip}(0,T; H^{s-1}(\mathbb{R}^3)),\ j = 1, \dots, \mathcal{N} \ \Big| \\ & \sup_{t\in[0,T]} \|\{w_j^\pm(t,\cdot)\}_{j\le\mathcal{N}}\|_{\mathbb{H}^s} := \sup_{t\in[0,T]} \sum_{j=1}^{\mathcal{N}} \|w_j^+(t,\cdot)\|_{H^s(\mathbb{R}^3)} + \|w_j^-(t,\cdot)\|_{H^s(\mathbb{R}^3)} \\ & \le \mathcal{K}\|\{v_{0j}^\pm\}_{j\le\mathcal{N}}\|_{\mathbb{H}^s} \Bigg\}, \end{aligned}$$

with $\mathcal{K} > 1$ a numerical constant. We then define the iteration map $\mathcal{F}$ as follows. For any sequence $\{w_j^\pm\}_{j\le\mathcal{N}} \in W_T$ the image $\mathcal{F}(\{w_j^\pm\}_{j\le\mathcal{N}})$ is the unique solution $\{v_j^\pm\}_{j\le\mathcal{N}}$ of

$$\partial_t v_j^\pm + u[\{w_j^\pm\}_{j\le\mathcal{N}}] \cdot \nabla_\perp v_j^\pm = 0, \tag{3.5}$$

with $v_{0j}^\pm$ as initial condition. We first show that $\mathcal{F}$ maps W_T onto itself for T small enough. If we apply the operator ∂^α to (3.5) for $|\alpha| \le s$ and take the L^2-scalar product with $\partial^\alpha v_j^\pm$ then we get

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha v_j^\pm\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} \partial^\alpha(u[\{w_j^\pm\}_{j\le\mathcal{N}}] \cdot \nabla_\perp v_j^\pm)\partial^\alpha v_j^\pm \, dx = 0. \tag{3.6}$$

Let us estimate the second term of (3.6). For $i = 1, 2$, using Leibniz rules, we have

$$\begin{aligned} & \int_{\mathbb{R}^3} \partial^\alpha(u_i[\{w_j^\pm\}_{j\le\mathcal{N}}]\partial_i v_j^\pm)\partial^\alpha v_j^\pm \, dx \\ & \quad = \int_{\mathbb{R}^3} \partial^\alpha v_j^\pm \sum_\beta \binom{\alpha}{\beta} \partial^\beta u_i[\{w_j^\pm\}_{j\le\mathcal{N}}]\partial^{\alpha-\beta}\partial_i v_j^\pm \, dx. \end{aligned} \tag{3.7}$$

The sum in (3.7) is made over all the terms with $\beta = \{\beta_i\}_{i=1}^3$, such that $0 \le \beta_i \le \alpha_i$ and the combination $\binom{\alpha}{\beta}$ are positive constant. Distinguishing the case $\beta = 0$ from others, equality (3.7) becomes for $i = 1, 2$,

$$\int_{\mathbb{R}^3} \partial^\alpha (u_i \partial_i v_j^\pm) \partial^\alpha v_j^\pm \, dx = \frac{1}{2} \binom{\alpha}{0} \int_{\mathbb{R}^3} u_i \partial_i (\partial^\alpha v_j^\pm)^2 \, dx + \int_{\mathbb{R}^3} \partial^\alpha v_j^\pm \sum_{\beta > 0} \binom{\alpha}{\beta} \partial^{\beta-\gamma} \partial^\gamma u_i \partial^{\alpha-\beta} \partial_i v_j^\pm \, dx, \quad (3.8)$$

with γ such that $|\gamma| = 1$. Using integration by parts the first term of the right-hand side of (3.8) can be estimated as

$$\frac{1}{2} \binom{\alpha}{0} \left| \int_{\mathbb{R}^3} u_i \partial_i (\partial^\alpha v_j^\pm)^2 \, dx \right| \leq C(\alpha) \|u_i\|_{W^{1,\infty}(\mathbb{R}^3)} \|v_j^\pm\|^2_{H^s(\mathbb{R}^3)}.$$

Using the Cauchy–Schwarz inequality and the interpolation inequality (see Proposition 3.6, Chap. 13 of Ref. 70)

$$\|\partial^{\beta-\gamma} \partial^\gamma f \partial^{\alpha-\beta} \partial_i g\|_{L^2(\mathbb{R}^3)} \leq C(s) (\|\partial^\gamma f\|_{L^\infty(\mathbb{R}^3)} \|g\|_{H^s(\mathbb{R}^3)} + \|f\|_{H^s(\mathbb{R}^3)} \|\partial_i g\|_{L^\infty(\mathbb{R}^3)}),$$

the second term of the right-hand side of (3.8) can be estimated as

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} \partial^\alpha v_j^\pm \sum_{\beta > 0} \binom{\alpha}{\beta} \partial^{\beta-\gamma} \partial^\gamma u_i \partial^{\alpha-\beta} \partial_i v_j^\pm \right| dx \\ &\quad \leq C(s) \|v_j^\pm\|^2_{H^s(\mathbb{R}^3)} \|\partial_i u_i\|_{L^\infty(\mathbb{R}^3)} + \|v_j^\pm\|_{H^s(\mathbb{R}^3)} \|\partial_i v_j^\pm\|_{L^\infty(\mathbb{R}^3)} \|u_i\|_{H^s(\mathbb{R}^3)} \\ &\quad \leq C(s) \|v_j^\pm\|^2_{H^s(\mathbb{R}^3)} \|u_i\|_{H^s(\mathbb{R}^3)}, \end{aligned} \quad (3.9)$$

where we have used the Sobolev imbedding $H^s(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$ for $s > n/2 + 1$, with $n = 3$. Gathering (3.6)–(3.9) we get

$$\frac{d}{dt} \|v_j^\pm\|_{H^s(\mathbb{R}^3)} \leq C(s) \|v_j^\pm\|_{H^s(\mathbb{R}^3)} (\|u_1\|_{H^s(\mathbb{R}^3)} + \|u_2\|_{H^s(\mathbb{R}^3)}), \quad j = 1, \ldots, \mathcal{N}. \quad (3.10)$$

Let us now estimate the term $\|u_i\|_{H^s(\mathbb{R}^3)}$. For $i = 1, 2$ we get

$$\begin{aligned} \left\| u_i \left[\{w_j^\pm\}_{j \leq \mathcal{N}} \right] \right\|^2_{H^s(\mathbb{R}^3)} &= \sum_{|\alpha| \leq s} \left\| \mathbb{K}_i \overset{\perp}{*} \partial^\alpha \sum_{j=1}^{\mathcal{N}} \mathcal{A}_j (w_j^+ - w_j^-) \right\|^2_{L^2(\mathbb{R}^3)} \\ &\leq A^2 \|\mathbb{K}_i\|^2_{L^1(\mathbb{R}^2)} \sum_{i=1}^{\mathcal{N}} \sum_{|\alpha| \leq s} (\|w_j^+\|^2_{H^\alpha(\mathbb{R}^3)} + \|w_j^-\|^2_{H^\alpha(\mathbb{R}^3)}) \\ &\leq A^2 \|\mathbb{K}_i\|^2_{L^1(\mathbb{R}^2)} \sum_{i=1}^{\mathcal{N}} (\|w_j^+\|^2_{H^s(\mathbb{R}^3)} + \|w_j^-\|^2_{H^s(\mathbb{R}^3)}) \\ &\leq A^2 \|\mathbb{K}_i\|^2_{L^1(\mathbb{R}^2)} \|\{w_j^\pm\}_{j \leq \mathcal{N}}\|^2_{\mathbb{H}^s}. \end{aligned} \quad (3.11)$$

Plugging (3.11) into (3.10), and summing over j in (3.10) we finally obtain the differential inequality

$$\frac{d}{dt}\|\{v_j^\pm(t)\}_{j\le\mathcal{N}}\|_{\mathbb{H}^s} \le C(s, A, \|\mathbb{K}\|_{L^1(\mathbb{R}^2)})\|\{w_j^\pm(t)\}_{j\le\mathcal{N}}\|_{\mathbb{H}^s}\|\{v_j^\pm(t)\}_{j\le\mathcal{N}}\|_{\mathbb{H}^s}. \qquad (3.12)$$

A Gronwall lemma then shows that $\|\{v_j^\pm(t)\}_{j\le\mathcal{N}}\|_{\mathbb{H}^s} \le \mathcal{K}\|\{v_{0j}^\pm\}_{j\le\mathcal{N}}\|_{\mathbb{H}^s}$ for all $t\in[0,T]$, T small enough. From (3.5) we have $v_j^\pm \in \mathrm{Lip}(0,T;H^{s-1}(\mathbb{R}^3))$ for $1\le j\le\mathcal{N}$. We then conclude that the application $\mathcal{F}$ maps W_T into itself. We now need to prove that $\mathcal{F}$ is a contraction. We consider two set of functions $\{w_j^{\pm,1}\}_{j\le\mathcal{N}}$ and $\{w_j^{\pm,2}\}_{j\le\mathcal{N}}$ belonging to W_T. We set $\{v_j^{\pm,1}\}_{j\le\mathcal{N}} := \mathcal{F}(\{w_j^{\pm,1}\}_{j\le\mathcal{N}})$, $\{v_j^{\pm,2}\}_{j\le\mathcal{N}} := \mathcal{F}(\{w_j^{\pm,2}\}_{j\le\mathcal{N}})$, $v_j^\pm = v_j^{\pm,1} - v_j^{\pm,2}$ and $w_j^\pm = w_j^{\pm,1} - w_j^{\pm,2}$ for all $1\le j\le\mathcal{N}$ and $u = u^1 - u^2$. The difference of Eq. (3.5) for $\{v_j^{\pm,1}\}$ and $\{v_j^{\pm,2}\}$ gives

$$\partial_t v_j^\pm + u\cdot\nabla_\perp v_j^{\pm,1} + u^2\cdot\nabla_\perp v_j^\pm = 0, \quad v_j^\pm(t=0) = 0. \qquad (3.13)$$

In the same manner we obtained (3.6), we deduce from (3.13)

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha v_j\|_{L^2(\Omega)} + \int_{\mathbb{R}^3}\partial^\alpha(u\cdot\nabla_\perp v_j^{\pm,1})\partial^\alpha v_j^\pm\,dx + \int_{\mathbb{R}^3}\partial^\alpha(u^2\cdot\nabla_\perp v_j^\pm)\partial^\alpha v_j^\pm\,dx = 0. \qquad (3.14)$$

Using the estimates of Proposition 3.7, Chap. 13 of Ref. 70 the second term of the left-hand side of (3.14) for $|\alpha|\le s-1$ is bounded as follows:

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\partial^\alpha(u\cdot\nabla_\perp v_j^{\pm,1})\partial^\alpha v_j^\pm\,dx\right| \\
&\quad\le \|\partial^\alpha v_j^\pm\|_{L^2(\mathbb{R}^3)}\|\partial^\alpha(u\cdot\nabla_\perp v_j^{\pm,1})\|_{L^2(\mathbb{R}^3)} \\
&\quad\le C(s)\|v_j^\pm\|_{H^{s-1}(\mathbb{R}^3)}(\|u\|_{L^\infty(\mathbb{R}^3)}\|v_j^{\pm,1}\|_{H^s(\mathbb{R}^3)} + \|u\|_{H^{s-1}(\mathbb{R}^3)}\|v_j^{\pm,1}\|_{W^{1,\infty}(\mathbb{R}^3)}) \\
&\quad\le C(s, A, \|\mathbb{K}\|_{L^1(\mathbb{R}^2)})\|v_j^\pm\|_{H^{s-1}(\mathbb{R}^3)}\|v_j^{\pm,1}\|_{H^s(\Omega)}\|\{w_j^\pm\}_{j\le\mathcal{N}}\|_{\mathbb{H}^{s-1}}.
\end{aligned}$$

For the second term of the left-hand side of (3.14) we proceed similarly to (3.9). Using the fact that $\nabla_\perp\cdot u^2 = 0$ and provided that $s > 5/2$ we get

$$\begin{aligned}
\left|\int_{\mathbb{R}^3}\partial^\alpha(u^2\cdot\nabla_\perp v_j^\pm)\partial^\alpha v_j^\pm\,dx\right| &\le \frac{1}{2}\binom{\alpha}{0}\left|\int_{\mathbb{R}^3}u^2\cdot\nabla_\perp(\partial^\alpha v_j^\pm)^2\,dx\right| \\
&\quad + \left|\int_{\mathbb{R}^3}\partial^\alpha v_j^\pm\sum_{\beta>0}\binom{\alpha}{\beta}\partial^{\beta-\gamma}\partial^\gamma u^2\cdot\partial^{\alpha-\beta}\nabla_\perp v_j^\pm\right|dx \\
&\le C(s)\|v_j^\pm\|^2_{H^{s-1}(\mathbb{R}^3)}\|u^2\|_{H^s(\mathbb{R}^3)} \\
&\le C(s, A, \|\mathbb{K}\|_{L^1(\mathbb{R}^2)})\|v_j^\pm\|^2_{H^{s-1}(\mathbb{R}^3)}\|\{w_j^{\pm,2}\}_{j\le\mathcal{N}}\|_{\mathbb{H}^s}.
\end{aligned}$$

Since $\|\{v_j^{\pm,1}(t)\}_{j\le\mathcal{N}}\|_{\mathbb{H}^m}$, $\|\{w_j^{\pm,2}(t)\}_{j\le\mathcal{N}}\|_{\mathbb{H}^m} \le \mathcal{K}\|\{v_{0j}^\pm\}_{j\le\mathcal{N}}\|_{\mathbb{H}^m}$, we finally obtain

$$\begin{aligned}
\frac{d}{dt}\|\{v_j^\pm(t)\}_{j\le\mathcal{N}}\|_{\mathbb{H}^{s-1}} &\le C(s, A, \mathcal{K}, \|\mathbb{K}\|_{L^1(\mathbb{R}^2)}, \|\{v_{0j}^\pm\}_{j\le\mathcal{N}}\|_{\mathbb{H}^m}) \\
&\quad\cdot(\|\{v_j^\pm(t)\}_{j\le\mathcal{N}}\|_{\mathbb{H}^{s-1}} + \|\{w_j^\pm(t)\}_{j\le\mathcal{N}}\|_{\mathbb{H}^{s-1}}).
\end{aligned}$$

Once again, a Gronwall lemma shows that $\mathcal{F}$ is a contraction provided that T is small enough. □

Remark 3.1. We can also prove the global existence of regular solution of the system (3.3), but it is not necessary for the further proof of regular solution for the gyro-water-bag model. The crucial ingredient for proving the global existence and uniqueness of classical (and weak) solution for the system (3.3) is the log-Lipschitz (or quasi-Lipschitz) estimates on the kernel $\mathbb{K}$ which can be effectively proved by using the properties of the modified Bessel function. Due to the log-Lipschitz condition on the kernel $\mathbb{K}$, we can then adapt the method developed for inviscid incompressible Euler equations[58,48,1,57] by using the Lagrangian representation of the contours which remain constant along the incompressible Lagrangian flow defining a volume preserving map.

3.2. *The longitudinal dynamic system*

In this section, we consider the initial value problem in $\mathbb{R}^3$,

$$\begin{aligned} &\partial_t v_j^\pm + v_j^\pm \partial_z v_j^\pm + \partial_z \phi = 0, \quad v_j^\pm(0,\cdot) = v_{0j}^\pm(\cdot), \quad j = 1,\dots,\mathcal{N}, \\ &-\Delta_\perp \phi + \phi = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_j (v_j^+ - v_j^-) - 1. \end{aligned} \tag{3.15}$$

Therefore we have the existence theorem.

Theorem 3.2. (Local classical solution) *Assume* $v_{0j}^\pm \in H^s(\mathbb{R}^3)$ *with* $m > n/2 + 1$, $n = 3$ *and* $\mathcal{A}_j$ *strictly positive real numbers*, $1 \le j \le \mathcal{N}$. *Then for all* $\mathcal{N}$ *there exists a time* $T > 0$ *that depends only on* $\|v_{0j}^\pm\|_{H^s(\mathbb{R}^3)}$, $\mathcal{N}$, *and* $A = \max_{j\le\mathcal{N}} |\mathcal{A}_j|$, *such that Eq.* (3.15) *has a unique solution*

$$v_j^\pm \in L^\infty(0,T; H^s(\mathbb{R}^3)) \cap \mathrm{Lip}(0,T; H^{s-1}(\mathbb{R}^3)), \quad j = 1,\dots,\mathcal{N}.$$

Proof. If we set $V = (v_1^+,\dots,v_{\mathcal{N}}^+, v_1^-,\dots,v_{\mathcal{N}}^-)^{\mathrm{T}}$ the system of equations (3.15) can be recast in the quasilinear system

$$\partial_t V + \mathrm{Op}(\mathcal{B}(V(t,x),\xi))V = 0, \tag{3.16}$$

where the pseudo-differential operator $\mathrm{Op}(q(t,x,\xi))$ of symbol $q(t,x,\xi) = \mathcal{B}(V(t,x), \xi)$ is defined by

$$\mathrm{Op}(q(t,x,\xi))\psi = \int_{\mathbb{R}^3} q(t,x,\xi)\mathcal{F}\psi(\xi)\exp(ix\cdot\xi)\,d\xi,$$

for every smooth function ψ, with $\mathcal{F}\psi$ the Fourier transform of ψ. The symbol $q(t,x,\xi)$ is defined as

$$q(t,x,\xi) = q_1(t,x,\xi) + q_2(t,x,\xi) = i\xi_z(\tilde{q}_1(t,x,\xi) + \tilde{q}_2(t,x,\xi)),$$

where

$$\tilde{q}_1(t,x,\xi) = \mathrm{diag}(V(t,x)), \quad \text{and} \quad \tilde{q}_2(t,x,\xi) = \frac{\mathbb{1}\mathcal{A}_{\#}^{\mathrm{T}}}{1+|\xi_\perp|^2},$$

with

$$\mathcal{A} = (\mathcal{A}_1,\dots,\mathcal{A}_{\mathcal{N}})^{\mathrm{T}}, \quad \mathcal{A}_{\#} = (\mathcal{A}^{\mathrm{T}}, -\mathcal{A}^{\mathrm{T}})^{\mathrm{T}}, \quad \text{and} \quad \mathbb{1} = (\underbrace{1,\dots,1}_{2\mathcal{N}\ \text{times}})^{\mathrm{T}}.$$

Let us first show that the matrix-symbol q has $2\mathcal{N}$ distinct purely imaginary eigenvalues. To this purpose it is equivalent to show that the matrix-symbol $\tilde{q}$ has $2\mathcal{N}$ distinct real eigenvalues. Let λ be a number; then after some rearrangement of the line of $\tilde{q} - \lambda\mathcal{I}$, the latter matrix take the form

$$\begin{pmatrix} v_1^+ - \lambda & & -v_2^+ + \lambda & & & & \\ & \ddots & & \ddots & & 0 & \\ & & v_{\mathcal{N}}^+ - \lambda & & -v_1^- + \lambda & & \\ & & & & v_1^- - \lambda & -v_2^- + \lambda & \\ & 0 & & & \ddots & \ddots & \\ & & & & & v_{\mathcal{N}-1}^- - \lambda & -v_{\mathcal{N}}^- + \lambda \\ \frac{\mathcal{A}_1}{1+|\xi_\perp|^2}, \dots, & \frac{\mathcal{A}_{\mathcal{N}}}{1+|\xi_\perp|^2}, & -\frac{\mathcal{A}_1}{1+|\xi_\perp|^2}, \dots, & -\frac{\mathcal{A}_{\mathcal{N}-1}}{1+|\xi_\perp|^2}, & & & v_{\mathcal{N}}^- - \lambda - \frac{\mathcal{A}_{\mathcal{N}}}{1+|\xi_\perp|^2} \end{pmatrix}. \tag{3.17}$$

If we take the determinant of (3.17) we get the polynomial of degree $2\mathcal{N}$

$$P_{2\mathcal{N}}(\lambda) = \prod_{j=1}^{\mathcal{N}} (v_j^+ - \lambda)(v_j^- - \lambda)\left(1 - \sum_{j=1}^{\mathcal{N}} \frac{n_j(1+|\xi_\perp|^2)^{-1}}{(v_j^+ - \lambda)(v_j^- - \lambda)}\right), \tag{3.18}$$

where $n_i = \mathcal{A}_j(v_j^+ - v_j^-) \geq 0$, $\mathcal{A}_j \geq 0$ and $\cdots < v_j^- < \cdots < v_1^- < v_1^+ < \cdots < v_j^+ < \cdots$. We then observe that

$$\mathrm{sign}(P_{2\mathcal{N}}(0)) = \begin{cases} \text{positive} & \text{if } \mathcal{N} \text{ even}, \\ \text{negative} & \text{if } \mathcal{N} \text{ odd} \end{cases}$$

and

$$\mathrm{sign}(P_{2\mathcal{N}}(v_j^{\pm})) = \begin{cases} (-1)^j & \text{if } \mathcal{N} \text{ odd}, \\ (-1)^{j+1} & \text{if } \mathcal{N} \text{ even}, \end{cases} \quad j = 1,\dots,\mathcal{N}.$$

Consequently the polynomial $P_{2\mathcal{N}}$ oscillates $2\mathcal{N}-2$ times around zero and has $2\mathcal{N}-2$ roots, $\mathcal{N}-1$ positive $\{\lambda_j^+\}_{1\leq j\leq\mathcal{N}-1}$, and $\mathcal{N}-1$ negative $\{\lambda_j^-\}_{1\leq j\leq\mathcal{N}-1}$. Therefore we can factorize $P_{2\mathcal{N}}$ as follows:

$$P_{2\mathcal{N}}(\lambda) = Q_{2\mathcal{N}-2}(\lambda)S_2(\lambda),$$

with $Q_{2\mathcal{N}-2}(\lambda) = \prod_{j=1}^{\mathcal{N}-1}(\lambda - \lambda_j^+)(\lambda - \lambda_j^-)$ and $S_2(\lambda) = \lambda^2 + a\lambda + b$. If $\mathcal{N}$ is even then $P_{2\mathcal{N}}(0) > 0$ and $Q_{2\mathcal{N}-2}(0) < 0$. Therefore $S_2(0) < 0$ and $S_2(\lambda)$ has two distinct real roots of opposite sign. If $\mathcal{N}$ is now odd then $P_{2\mathcal{N}}(0) < 0$ and $Q_{2\mathcal{N}-2}(0) > 0$. Therefore $S_2(0) < 0$ and $S_2(\lambda)$ has again two distinct real roots of opposite sign. Finally we conclude that $P_{2\mathcal{N}}(\lambda)$ has $2\mathcal{N}$ distinct real roots, $\mathcal{N}$ positive and $\mathcal{N}$ negative. Since $P_{2\mathcal{N}}(\lambda) \sim \lambda^{2\mathcal{N}} > 0$ when $\lambda \sim \pm\infty$ and as $P_{2\mathcal{N}}(v_{\mathcal{N}}^{\pm}) < 0$ when $\mathcal{N}$ is even or odd therefore we have $\pm v_{\mathcal{N}}^{\pm} < \pm\lambda_{\mathcal{N}}^{\pm} < \infty$. Therefore the $2\mathcal{N} - 2$ other eigenvalues are such that $\pm v_j^{\pm} < \pm\lambda_j^{\pm} < \pm v_{j+1}^{\pm}$. Therefore the matrix-symbol q is diagonalizable and has distinct purely imaginary eigenvalues $i\lambda_\nu(V,\xi)$, smooth in V and ξ such that $\lambda_1(V,\xi) < \cdots < \lambda_j(V,\xi) < \cdots < \lambda_{2\mathcal{N}}(V,\xi)$. Therefore the system (3.15) is strictly hyperbolic. Instead of building a symbolic symmetrizer by spectral projections onto the $\lambda_\nu(V,\xi)$-eigenspaces of $\mathcal{B}$ (see Refs. 49, 51 and 69–71) thanks to the Dunford formula (a Cauchy integral formula-type[29]) and spectral separation (cf. Theorem 6, Chap. 17 of Ref. 50), we can directly construct the symbolic symmetrizer by finding an entropy of the transverse system. We will see below that the energy will supply a convex entropy. If we set

$$e_j = \frac{\mathcal{A}_j}{3}\left(v_j^{+^3} - v_j^{-^3}\right) = n_j u_j^2 + \frac{n_j^3}{12\mathcal{A}_j},$$

and use Eqs. (2.8)–(2.9) without the transverse terms, we obtain for all $j \in [1, \mathcal{N}]$

$$\partial_t\left(\frac{e_j}{2}\right) + \partial_z\left(u_j\frac{e_j}{2} + u_j\frac{n_j^3}{12\mathcal{A}_j^2}\right) + n_j u_j \partial_z\phi = 0. \tag{3.19}$$

Summing over all the bags, and using the continuity equation we obtain from Eq. (3.19)

$$\begin{aligned}\partial_t\left(\sum_{j=1}^{\mathcal{N}}\frac{e_j}{2}\right) &+ \partial_z\left(\sum_{j=1}^{\mathcal{N}} u_j\frac{e_j}{2} + u_j\frac{n_j^3}{12\mathcal{A}_j^2}\right)\\ &= -\partial_z\phi\sum_{j=1}^{\mathcal{N}} n_j u_j = \partial_z\left(\phi\sum_{j=1}^{\mathcal{N}} n_j u_j\right) + \phi\partial_t\sum_{j=1}^{\mathcal{N}} n_j. \end{aligned} \tag{3.20}$$

Using the quasi-neutrality equation (3.2) and integration by parts, the second term of the right-hand side of Eq. (3.20) becomes

$$\int_{\mathbb{R}^2} dx_\perp \phi\partial_t\sum_{j=1}^{\mathcal{N}} n_j = \frac{1}{2}\partial_t\int_{\mathbb{R}^2} dx_\perp(|\nabla_\perp\phi|^2 + |\phi|^2),$$

with $\phi = G \overset{\perp}{*} \sum_{j=1}^{\mathcal{N}} n_j$. Therefore the longitudinal system conserved the total energy

$$\frac{1}{2}\int_{\mathbb{R}^3} dx\left(\sum_{j=1}^{\mathcal{N}} e_j + |\nabla_\perp\phi|^2 + |\phi|^2\right). \tag{3.21}$$

If we now drop the term corresponding to the transverse gradient of the electric potential in the energy density (the integrand of (3.21)), which means that we remove the polarization term (the transverse Laplacian operator) in the quasi-neutrality equation, we obtain the entropy

$$\eta(V) = \frac{1}{2}\left(\sum_{j=1}^{\mathcal{N}} n_j\right)^2 + \sum_{j=1}^{\mathcal{N}} \frac{e_j}{2},$$

and its Hessian

$$\nabla^2\eta(V) = D(\mathcal{A}_\#)D(V) + \mathcal{A}_\#\mathcal{A}_\#^{\mathrm{T}}, \tag{3.22}$$

where $D(V) = \mathrm{diag}(V(t,x))$ and $D(\mathcal{A}_\#) = \mathrm{diag}(\mathcal{A}_\#)$. Using (3.22) we define

$$S(V,\xi) = D(\mathcal{A}_\#)D(V) + \frac{\mathcal{A}_\#\mathcal{A}_\#^{\mathrm{T}}}{1+|\xi_\perp|^2} = S_1 + S_2.$$

We obviously observe that S is a Hermitian matrix and that Sq is a skew-Hermitian matrix. Moreover the operator $\mathrm{Op}(S)$ is Hermitian, i.e. $\mathrm{Op}(S) = \mathrm{Op}(S)^\star$, since it is easily verified by a direct check that $\mathrm{Op}(S_i) = \mathrm{Op}(S_i)^\star$ for $i = 1, 2$. Therefore the operator $\mathrm{Op}(S)$ will be a good candidate for the symmetrizer. Let us note that $S_1 \in \mathscr{C}^1 S^0_{\mathrm{cl}} \cap H^s S^0_{\mathrm{cl}}$ as long as $V \in \mathscr{C}^1 \cap H^s$. Let us now obtain *a priori* estimates. We now set

$$Q = \mathrm{Op}(S) + \kappa\Lambda^{-1}, \tag{3.23}$$

with the definition $\Lambda^s = (1-\Delta)^{s/2}$ and where $(\cdot)^\star$ denotes the transconjugate of a matrix or the dual of an operator. The constant $\kappa > 0$ is chosen such that Q is a positive definite operator on L^2; hence invertible, since for Hermitian operator the origin is an isolated point of the spectrum of finite multiplicity.[51] In other words, it means that there exists a constant $c_0 > 0$ such that $\langle Q\psi, \psi\rangle \geq c_0\|\psi\|^2_{L^2(\mathbb{R}^3)}$ where $\langle\cdot,\cdot\rangle$ stands for the L^2-Hermitian scalar product. Let us notice that $\|\Lambda^s \cdot\|_{L^2}$ defines a norm which is equivalent to the H^s-norm. We aim to estimate $\|\Lambda^s V\|_{L^2(\mathbb{R}^3)}$. Let us note first that

$$\partial_t\langle Q\Lambda^s V, \Lambda^s V\rangle = \langle \partial_t Q\Lambda^s V, \Lambda^s V\rangle + 2\Re e\langle Q\partial_t\Lambda^s V, \Lambda^s V\rangle. \tag{3.24}$$

Let us first estimate the first term of the right-hand side of (3.24):

$$\begin{aligned}\langle \partial_t Q\Lambda^s V, \Lambda^s V\rangle &\leq |\langle \mathrm{Op}(\partial_t S)\Lambda^s V, \Lambda^s V\rangle| \\ &\leq \|\mathrm{Op}(\partial_t S)\Lambda^s V\|_{L^2(\mathbb{R}^3)}\|\Lambda^s V\|_{L^2(\mathbb{R}^3)} \\ &\leq C(\|\partial_t V\|_{\mathscr{C}(\mathbb{R}^3)})\|\Lambda^s V\|^2_{L^2(\mathbb{R}^3)} \\ &\leq C(\|V\|_{\mathscr{C}^1(\mathbb{R}^3)})\|\Lambda^s V\|^2_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{3.25}$$

Using Eq. (3.16), we get the following decomposition for the second term of the right-hand side of (3.24):

$$\begin{aligned}Q\partial_t\Lambda^s V &= -Q\Lambda^s\mathrm{Op}(q)V \\ &= -Q\mathrm{Op}(q)\Lambda^s V + Q[\mathrm{Op}(q), \Lambda^s]V.\end{aligned} \tag{3.26}$$

Let us first estimate the commutator $[\mathrm{Op}(q),\Lambda^s]$ in the second term of the right-hand side of (3.26). Since the commutator can be decomposed as $[\mathrm{Op}(q),\Lambda^s] = [\mathrm{Op}(q_1),\Lambda^s] + [\mathrm{Op}(q_2),\Lambda^s]$ and $[\mathrm{Op}(q_2),\Lambda^s] = [\mathbb{1}\mathcal{A}_{\#}^{\mathrm{T}}\partial_z\Lambda_{\perp}^{-2},\Lambda^s] = 0$ where $\Lambda_{\perp}^{s} = (1-\Delta_{\perp})^{s/2}$ with $s\in\mathbb{R}$, then it remains to estimate $[\mathrm{Op}(q_1),\Lambda^s]$. Since the differential operator $\Lambda^s \in \mathrm{Op}\, S_{1,0}^{s}$ and the symbol $S_1 \in \mathscr{C}^1 S_{\mathrm{cl}}^0 \cap H^s S_{\mathrm{cl}}^0$, using the Kato–Ponce estimate 3.6.1, Chap. 3 of Ref. 71 or its generalization (for pseudo-differential operator with symbol in $\mathscr{C}^1 S_{\mathrm{cl}}^0 \cap H^s S_{\mathrm{cl}}^0$) given by the Proposition 4.1.F, Chap. 4 of Ref. 71, we have the estimate

$$\begin{aligned} &\|[\mathrm{Op}(q_1),\Lambda^s]\psi\|_{L^2(\mathbb{R}^3)} \\ &\quad \le C(\|\psi\|_{H^s(\mathbb{R}^3)}\|\partial_z V\|_{L^\infty(\mathbb{R}^3)} + \|\psi\|_{\mathrm{Lip}(\mathbb{R}^3)}\|\partial_z V\|_{H^{s-1}(\mathbb{R}^3)}). \end{aligned} \tag{3.27}$$

Moreover we claim that we have

$$\|Q\psi\|_{L^2(\mathbb{R}^3)} \le C(\|V\|_{\mathscr{C}(\mathbb{R}^3)})\|\psi\|_{L^2(\mathbb{R}^3)} \le c_1\|\psi\|_{L^2(\mathbb{R}^3)}. \tag{3.28}$$

Indeed, using the decomposition (3.23) we get

$$\begin{aligned} \|Q\psi\|_{L^2(\mathbb{R}^3)} &\le \|\mathrm{Op}(S)\psi\|_{L^2(\mathbb{R}^3)} + \|\kappa\Lambda^{-1}\psi\|_{L^2(\mathbb{R}^3)} \\ &\le C(\|V\|_{\mathscr{C}(\mathbb{R}^3)})\|\psi\|_{L^2(\mathbb{R}^3)} + C\|\psi\|_{L^2(\mathbb{R}^3)} \\ &\le C(\|V\|_{\mathscr{C}(\mathbb{R}^3)})\|\psi\|_{L^2(\mathbb{R}^3)}, \end{aligned}$$

which proves estimate (3.28). Now using estimate (3.27) and (3.28) we obtain

$$\begin{aligned} \langle Q[\mathrm{Op}(q),\Lambda^s]V,\Lambda^s V\rangle &\le \|Q[\mathrm{Op}(q),\Lambda^s]V\|_{L^2(\mathbb{R}^3)}\|\Lambda^s V\|_{L^2(\mathbb{R}^3)} \\ &\le C(\|V\|_{\mathscr{C}(\mathbb{R}^3)})\|[\mathrm{Op}(q),\Lambda^s]V\|_{L^2(\mathbb{R}^3)}\|\Lambda^s V\|_{L^2(\mathbb{R}^3)} \\ &\le C(\|V\|_{\mathscr{C}(\mathbb{R}^3)})\|\Lambda^s V\|_{L^2(\mathbb{R}^3)} \\ &\qquad \cdot\{\|\partial_z V\|_{L^\infty(\mathbb{R}^3)}\|V\|_{H^s(\mathbb{R}^3)} + \|\partial_z V\|_{H^{s-1}(\mathbb{R}^3)}\|V\|_{\mathrm{Lip}(\mathbb{R}^3)}\} \\ &\le C(\|V\|_{\mathscr{C}^1(\mathbb{R}^3)})\|V\|^2_{H^s(\mathbb{R}^3)}. \end{aligned} \tag{3.29}$$

Let us now estimate the first term of the right-hand side of (3.26). We first observe that

$$\langle Q\mathrm{Op}(q)\Lambda^s V,\Lambda^s V\rangle = \langle \mathrm{Op}(S)\mathrm{Op}(q)\Lambda^s V,\Lambda^s V\rangle + \langle \kappa\Lambda^{-1}\mathrm{Op}(q)\Lambda^s V,\Lambda^s V\rangle. \tag{3.30}$$

The second term of the right-hand side of (3.30) can be bounded as

$$\begin{aligned} \langle \kappa\Lambda^{-1}\mathrm{Op}(q)\Lambda^s V,\Lambda^s V\rangle &\le C\|\Lambda^{-1}\mathrm{Op}(q)\Lambda^s V\|^2_{L^2(\mathbb{R}^3)}\|\Lambda^s V\|^2_{L^2(\mathbb{R}^3)} \\ &\le C\|\mathrm{Op}(q)\Lambda^s V\|^2_{H^{-1}(\mathbb{R}^3)}\|\Lambda^s V\|^2_{L^2(\mathbb{R}^3)} \\ &\le C(\|V\|_{\mathscr{C}(\mathbb{R}^3)})\|V\|^2_{H^s(\mathbb{R}^3)}. \end{aligned} \tag{3.31}$$

To get a bound on the first term of the right-hand side of (3.30), we can proceed as follows. After a little algebra, we have

$$\begin{aligned} \mathrm{Op}(S)\mathrm{Op}(q) &= \sum_{i,j=1}^{2}\mathrm{Op}(S_i)\mathrm{Op}(q_j) \\ &= D(\mathcal{A}_{\#})D(V)^2\partial_z + \mathcal{A}_{\#}\mathcal{A}_{\#}^{\mathrm{T}}D(V)\Lambda_{\perp}^{-2}\partial_z + \mathcal{A}_{\#}\mathcal{A}_{\#}^{\mathrm{T}}\Lambda_{\perp}^{-2}D(V)\partial_z, \end{aligned} \tag{3.32}$$

while

$$\begin{aligned}\mathrm{Op}(Sq) &= \mathrm{Op}\left(\sum_{i,j=1}^{2} S_i q_j\right) = D(\mathcal{A}_\#)D(V)^2\partial_z + 2\mathcal{A}_\#\mathcal{A}_\#^{\mathrm{T}} D(V)\Lambda_\perp^{-2}\partial_z \\ &= \mathrm{Op}(S)\mathrm{Op}(q) - \mathcal{R},\end{aligned} \tag{3.33}$$

where the differential operator $\mathcal{R}$ and its dual are given by

$$\mathcal{R} = \mathcal{A}_\#\mathcal{A}_\#^{\mathrm{T}}[\Lambda_\perp^{-2}, D(V)]\partial_z \quad \text{and} \quad \mathcal{R}^\star = \mathcal{R} + \mathcal{A}_\#\mathcal{A}_\#^{\mathrm{T}}[\Lambda_\perp^{-2}, D(\partial_z V)]. \tag{3.34}$$

A direct computation of the dual operator of $\mathrm{Op}(Sq)$ gives

$$\mathrm{Op}(Sq)^\star = \mathrm{Op}((Sq)^\star) - 2(\mathcal{R} + D(\mathcal{A}_\#)D(V)D(\partial_z V) + \mathcal{A}_\#\mathcal{A}_\#^{\mathrm{T}}\Lambda_\perp^{-2}D(\partial_z V)). \tag{3.35}$$

Now using (3.32)–(3.35), and the fact that Sq is skew-Hermitian, we get

$$\begin{aligned}&2\Re e\langle \mathrm{Op}(S)\mathrm{Op}(q)\Lambda^s V, \Lambda^s V\rangle \\ &\quad = \langle(\mathrm{Op}(Sq) + \mathrm{Op}(Sq)^\star)\Lambda^s V, \Lambda^s V\rangle \ + \ \langle(\mathcal{R} + \mathcal{R}^\star)\Lambda^s V, \Lambda^s V\rangle \\ &\quad = -\langle(2D(\mathcal{A}_\#)D(V)D(\partial_z V) \ + \ 2\mathcal{A}_\#\mathcal{A}_\#^{\mathrm{T}}\Lambda_\perp^{-2}D(\partial_z V) \\ &\qquad + \mathcal{A}_\#\mathcal{A}_\#^{\mathrm{T}}[D(\partial_z V), \Lambda_\perp^{-2}])\Lambda^s V, \Lambda^s V\rangle.\end{aligned} \tag{3.36}$$

From (3.36) we get

$$2\Re e\langle \mathrm{Op}(S)\mathrm{Op}(q)\Lambda^s V, \Lambda^s V\rangle \le C(\|V\|_{\mathscr{C}^1(\mathbb{R}^3)})\|V\|_{H^s(\mathbb{R}^3)}^2. \tag{3.37}$$

Using expressions (3.24), (3.26) and (3.30) and gathering estimates (3.25), (3.29), (3.31) and (3.37) we finally obtain

$$\frac{d}{dt}\langle Q\Lambda^s V, \Lambda^s V\rangle \le C_2(\|V\|_{\mathscr{C}^1(\mathbb{R}^3)})\|V\|_{H^s(\mathbb{R}^3)}^2. \tag{3.38}$$

Therefore integrating the differential inequality (3.38) between time zero and t, using the property $\langle Q\psi, \psi\rangle \ge c_0\|\psi\|_{L^2(\mathbb{R}^3)}^2$ and estimate (3.28), a Gronwall lemma and Eq. (3.16) conclude that we have

$$V \in L^\infty(0, T; H^s(\mathbb{R}^3)) \cap \mathrm{Lip}(0, T; H^{s-1}(\mathbb{R}^3)). \tag{3.39}$$

The estimate (3.39) can just give a weak convergence of the solution sequence of a regularization of Eq. (3.16) since compact Sobolev embeddings failed in the whole space. Therefore we will obtain strong convergence of solution sequence in $L^\infty(0, T; L^2(\mathbb{R}^3))$ by proving that this sequence is a Cauchy sequence. Let us now consider the sequence of following regularized problem:

$$Q(V^k)\partial_t V^{k+1} + Q(V^k)\mathrm{Op}(\mathcal{B}(V^k, \xi))V^{k+1}, \quad V^{k+1}(t=0) = \rho_{\delta_{k+1}} * V_0, \tag{3.40}$$

where the mollifier $\rho_\delta = \rho(x/\delta)/\delta^3$ ($0 < \delta < 1$, $\int \rho dx = 1$) has the following property:

$$\|\partial_\delta \rho_\delta * \psi\|_{H^s(\mathbb{R}^3)} \le C\delta^{r-s-1}\|\psi\|_{H^r(\mathbb{R}^3)} \quad \forall\, r,\ s. \tag{3.41}$$

Following the proof of the estimate (3.39) we find that

$$V^k \in L^\infty(0, T; H^s(\mathbb{R}^3)) \cap \mathrm{Lip}(0, T; H^{s-1}(\mathbb{R}^3)). \tag{3.42}$$

Therefore there exists a subsequence still noted $\{V^k\}$ such that

$$V^k \rightharpoonup \overline{V} \quad \textit{weakly} \text{ in } \quad L^\infty(0, T; H^s(\mathbb{R}^3)) \cap \mathrm{Lip}(0, T; H^{s-1}(\mathbb{R}^3)).$$

By subtraction of Eq. (3.40) we obtain

$$\begin{aligned} &Q(V^k)\partial_t(V^{k+1} - V^k) + Q(V^k)\mathrm{Op}(\mathcal{B}(V^k, \xi))(V^{k+1} - V^k) \\ &\quad = F(V^k, V^{k-1}) \\ &\quad = -[Q(V^k) - Q(V^{k-1})]\partial_t V^k \\ &\qquad - [Q(V^k)\mathrm{Op}(\mathcal{B}(V^k, \xi)) - Q(V^{k-1})\mathrm{Op}(\mathcal{B}(V^{k-1}, \xi))] V^k, \\ &(V^{k+1} - V^k)(t = 0) = (\rho_{\delta_{k+1}} - \rho_{\delta_k}) * V_0. \end{aligned} \tag{3.43}$$

Multiplying Eq. (3.43) by $(V^{k+1} - V^k)$ and integrating on the whole space, using estimate (3.42), following the energy estimate procedure leading to (3.39) and observing the fact that

$$\begin{aligned} &\langle F(V^k(t), V^{k-1}(t)), V^{k+1}(t) - V^k(t)\rangle \\ &\quad \le C(\|V^k\|_{L^\infty(0,T;H^s(\mathbb{R}^3))}, \|V^k\|_{\mathrm{Lip}(0,T;H^{s-1}(\mathbb{R}^3))}) \\ &\qquad \cdot \|V^{k+1}(t) - V^k(t)\|_{L^2(\mathbb{R}^3)}\|V^k(t) - V^{k-1}(t)\|_{L^2(\mathbb{R}^3)}, \end{aligned}$$

we obtain for T small enough the estimate

$$\begin{aligned} &\|V^{k+1} - V^k\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \\ &\quad \le C\|(\rho_{\delta_{k+1}} - \rho_{\delta_k}) * V_0\|_{L^2(\mathbb{R}^3))} + CT\|V^k - V^{k-1}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}. \end{aligned}$$

Using (3.41) we obtain

$$\epsilon_k = \|(\rho_{\delta_{k+1}} - \rho_{\delta_k}) * V_0\|_{L^2(\mathbb{R}^3))} \le C\delta_k^{s-1}|\delta_{k+1} - \delta_k|\|V_0\|_{H^s(\mathbb{R}^3)}.$$

and it results that any good choice of δ_k (for example, $\delta_k = 1/k$) makes the series $\sum_k \epsilon_k$ convergent. Consequently if T is small enough there exists a constant $C < 1$ such that

$$\|V^{k+1} - V^k\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \le C\|V^k - V^{k-1}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \epsilon_k,$$

which proves that $\|V^{k+1} - V^k\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}$ is bounded for any k. Therefore we obtain

$$\|V^p - V^q\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \le \sum_{k=q+1}^{p} \|V^k - V^{k-1}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \le C|p - q|,$$

which proves that the sequence $\{V^k\}$ is a Cauchy sequence which has a strong limit point in $L^\infty(0, T; L^2(\mathbb{R}^3))$. Since $V^k(t, \cdot)$ is bounded in H^s and strongly converge in L^2 toward $V(t, \cdot)$ we have in fact $V \in L^\infty(0, T; H^s(\mathbb{R}^3))$. Indeed following classical argument there exists a subsequence still noted $\{V^k(t, \cdot)\}$ weakly convergent in H^s toward $\overline{V}(t, \cdot) \in H^s(\mathbb{R}^3)$. As the limit in $\mathcal{D}'$ is unique we have $\overline{V} = V$. From Eq. (3.40) we have also $V \in \mathrm{Lip}(0, T; H^{s-1}(\mathbb{R}^3))$. Since $V^k \to V$ strongly in $L^\infty(0, T; L^2(\mathbb{R}^3))$, we also have $Q(V^k) \to Q(V)$ and $\mathcal{B}(V^k, \xi) \to \mathcal{B}(V, \xi)$ strongly in $L^\infty(0, T; L^2(\mathbb{R}^3))$. As $V^k \rightharpoonup V$ weakly in $L^\infty(0, T; H^s(\mathbb{R}^3))$ we have

$$Q(V^k)\mathrm{Op}(\mathcal{B}(V^k, \xi))\, V^{k+1} \rightharpoonup Q(V)\mathrm{Op}(\mathcal{B}(V, \xi))\, V, \qquad \text{weakly-}* \text{ in } L^\infty(0, T; L^\infty(\mathbb{R}^3)).$$

As $\partial_t V^{k+1} \rightharpoonup \partial_t V$ weakly-$*$ in $L^\infty(0, T; L^\infty(\mathbb{R}^3))$ we have

$$Q(V^k)\partial_t V^{k+1} \rightharpoonup Q(V)\partial_t V, \quad \text{weakly-}* \text{ in } L^\infty(0, T; L^\infty(\mathbb{R}^3)),$$

which means that the limit point V satisfies Eq. (3.15). □

3.3. *The gyro-water-bag model*

In this section, we consider the initial value problem in $\mathbb{R}^3$,

$$\partial_t v_j^\pm - \nabla_\perp^\perp \phi \cdot \nabla_\perp v_j^\pm + v_j^\pm \partial_z v_j^\pm + \partial_z \phi = 0, \quad v_j^\pm(0, \cdot) = v_{0j}^\pm(\cdot), \quad j = 1, \dots, \mathcal{N},$$
$$-\Delta_\perp \phi + \phi = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_j(v_j^+ - v_j^-) - 1. \tag{3.44}$$

Therefore we have the existence theorem.

Theorem 3.3. (Local classical solution) *Assume $v_{0j}^\pm \in H^s(\mathbb{R}^3)$ with $s > n/2 + 1$, $n = 3$ and $\mathcal{A}_j$ strictly positive real numbers, $1 \le j \le \mathcal{N}$. Then for all $\mathcal{N}$ there exists a time $T > 0$ that depends only on $\|v_{0j}^\pm\|_{H^s(\mathbb{R}^3)}$, $\mathcal{N}$, $A = \max_{j \le \mathcal{N}} |\mathcal{A}_j|$, such that Eq. (3.44) has a unique solution*

$$v_j^\pm \in L^\infty(0, T; H^s(\mathbb{R}^3)) \cap \mathrm{Lip}(0, T; H^{s-1}(\mathbb{R}^3)) \cap \mathscr{C}(0, T; H^{s-1}(\mathbb{R}^3)), \quad j = 1, \dots, \mathcal{N}.$$

Proof. The idea of the proof is to construct an approximate solution sequence of the problem (3.44) by splitting the global evolution operator $\mathbb{S}$ associated to the transport equation (3.44) into the longitudinal evolution operator $\mathbb{S}_\mathcal{L}$ associated to the transport equation (3.15) and the transversal evolution operator $\mathbb{S}_\mathcal{T}$ associated to the transport equation (3.3). If we set $\Delta t = T/N$, and $t^n = n\Delta t$, then our construction can be summarized as

$$\begin{aligned} \widehat{V}_{n+1}(t^{n+1}) &= \mathbb{S}(t^n, t^{n+1})\, \widehat{V}_n(t^n) \\ &= \mathbb{S}_\mathcal{T}(t^n, t^{n+1}) \circ \mathbb{S}_\mathcal{L}(t^n, t^{n+1})\, \widehat{V}_n(t^n) \\ &= \mathbb{S}_\mathcal{T}(t^n, t^{n+1})\, \widetilde{V}_{n+1}(t^{n+1}), \end{aligned} \tag{3.45}$$

where $\widehat{V}_{n+1}(t)$ (respectively $\widetilde{V}_{n+1}(t)$) is the solution of Eq. (3.3) (respectively (3.15)) in the time interval $[t^n, t^{n+1}]$ with the initial condition $\widehat{V}_{n+1}(t^n) = \widetilde{V}_{n+1}(t^{n+1})$ (respectively, $\widetilde{V}_{n+1}(t^n) = \widehat{V}_n(t^n)$). Thus for $t \le T$ we define

$$\widetilde{V}^N(t) = \sum_{n=0}^{N-1} \widetilde{V}_{n+1}(t)\chi_{n+1}(t), \quad \widehat{V}^N(t) = \sum_{n=0}^{N-1} \widehat{V}_{n+1}(t)\chi_{n+1}(t),$$

with the function $\chi_{n+1}(t)$ equal to one on $]t^n, t^{n+1}]$ and zero elsewhere. From Sec. 3.1 (respectively, Sec. 3.2) we know that for Δt small enough there exists a unique regular solution $\widehat{V}_{n+1}(t)$ (respectively, $\widetilde{V}_{n+1}(t)$) on the interval $[t^n, t^{n+1}]$ launched by the initial condition $\widetilde{V}_{n+1}(t^{n+1})$ (respectively, $\widehat{V}_n(t^n)$). On the one hand using (3.38) we have

$$\langle Q\Lambda^s \widetilde{V}^N(t^{n+1}), \Lambda^s \widetilde{V}^N(t^{n+1})\rangle \le \langle Q\Lambda^s \widehat{V}^N(t^n), \Lambda^s \widehat{V}^N(t^n)\rangle + \int_{t^n}^{t^{n+1}} C_2(\|\widetilde{V}^N(\tau)\|_{\mathscr{C}^1(\mathbb{R}^3)})\|\widetilde{V}^N(\tau)\|^2_{H^s(\mathbb{R}^3)}\, d\tau. \quad (3.46)$$

On the other hand, applying the operator Q to Eq. (3.3), recasted as a system, and following energy estimate procedure leading to the differential inequality (3.12) we obtain

$$\begin{aligned} &\langle Q\Lambda^s \widehat{V}^N(t^{n+1}), \Lambda^s \widehat{V}^N(t^{n+1})\rangle \\ &\quad \le \langle Q\Lambda^s \widetilde{V}^N(t^{n+1}), \Lambda^s \widetilde{V}^N(t^{n+1})\rangle \\ &\qquad + \int_{t^n}^{t^{n+1}} C_1(\|\widetilde{V}^N(\tau)\|_{\mathscr{C}^1(\mathbb{R}^3)}, \|\widehat{V}^N(\tau)\|_{\mathscr{C}^1(\mathbb{R}^3)})\|\widehat{V}^N(\tau)\|^3_{H^s(\mathbb{R}^3)}\, d\tau. \end{aligned} \quad (3.47)$$

Let us set

$$\Theta^N(t) = \|\widetilde{V}^N(t)\|^2_{H^s(\mathbb{R}^3)} + \|\widehat{V}^N(t)\|^2_{H^s(\mathbb{R}^3)},$$

and

$$\Pi^N(t) = \langle Q\Lambda^s \widetilde{V}^N(t), \Lambda^s \widetilde{V}^N(t)\rangle + \langle Q\Lambda^s \widehat{V}^N(t), \Lambda^s \widehat{V}^N(t)\rangle.$$

If we combine both estimates (3.46) and (3.47) and sum over n we obtain

$$\Pi^N(t^N) \le \Pi^N(t^0) + \int_0^T C(\|\widetilde{V}^N(\tau)\|_{\mathscr{C}^1(\mathbb{R}^3)}, \|\widehat{V}^N(\tau)\|_{\mathscr{C}^1(\mathbb{R}^3)})(\Theta^N(t))^{3/2}\, dt. \quad (3.48)$$

Using inequality (3.48), and the following estimate:

$$c_0\|\psi\|_{L^2(\mathbb{R}^3)} \le \langle Q\psi, \psi\rangle \le c_1\|\psi\|_{L^2(\mathbb{R}^3)},$$

a Gronwall lemma implies that there exists a time $T > 0$ such that the sequences $\{\widehat{V}^N\}$ and $\{\widetilde{V}^N\}$ have a weak limit point $V^\dagger$ in the space $L^\infty(0, T; H^s(\mathbb{R}^3)) \cap \mathrm{Lip}(0, T; H^{s-1}(\mathbb{R}^3))$. The estimate (3.48) can just give a weak convergence of the solution sequences $\{\widehat{V}^N\}$ and $\{\widetilde{V}^N\}$ since compact Sobolev embeddings failed in the whole space. Let Ω be a compact subset of $\mathbb{R}^3$ with smooth boundary. As $\partial_t \widehat{V}^N$ and $\partial_t \widetilde{V}^N$ remain in a bounded set of $L^\infty(0, T; H^{s-1}(\mathbb{R}^3))$, then for $t, t' > 0$, and for all N we get

$$\|\widehat{V}^N(t) - \widehat{V}^N(t')\|_{H^{s-1}(\Omega)} \le C|t - t'| \quad \text{and} \quad \|\widetilde{V}^N(t) - \widetilde{V}^N(t')\|_{H^{s-1}(\Omega)} \le C|t - t'|.$$

Using the interpolation inequality

$$\|f\|_{H^\nu(\Omega)} \le C\|f\|^{1-\sigma}_{H^s(\Omega)}\|f\|^{\sigma}_{H^{s-1}(\Omega)},$$

with $\sigma \in (0,1)$ and $\nu = \sigma s + (1-\sigma)(s-1)$, and since $\widehat{V}^N$ and $\widetilde{V}^N$ belongs to $L^\infty(0,T;H^s(\Omega)) \cap \mathrm{Lip}(0,T;H^{s-1}(\Omega))$, then the sequences $\{\widehat{V}^N\}$ and $\{\widetilde{V}^N\}$ are bounded in $\mathscr{C}^\sigma(0,T;H^\nu(\Omega))$. As the embedding $H^{s+\varepsilon}(\Omega) \hookrightarrow H^s(\Omega)$ is compact, with $\varepsilon > 0$, then by Ascoli theorem the sequences $\{\widehat{V}^N\}$ and $\{\widetilde{V}^N\}$ are compact in $\mathscr{C}(0,T;H^{s-1}(\Omega))$. Then we can extract from sequences $\{\widehat{V}^N\}$ and $\{\widetilde{V}^N\}$ subsequences still denoted by $\{\widehat{V}^N\}$ and $\{\widetilde{V}^N\}$ such that

$$\widehat{V}^N \to V^\dagger \quad \text{in } \mathscr{C}(0,T;H^{s-1}(\Omega)),$$
$$\widetilde{V}^N \to V^\ddagger \quad \text{in } \mathscr{C}(0,T;H^{s-1}(\Omega)).$$

Let $\{\Omega_k\}$ be a countable increasing sequence of compact subsets of $\mathbb{R}^3$, with smooth boundary which cover $\mathbb{R}^3$. Then for each k successively, using the previous compactness result, we can extract from sequences $\{\widehat{V}^{N(k)}\}$ and $\{\widetilde{V}^{N(k)}\}$, subsequences which converge in $\mathscr{C}(0,T;H^{s-1}(\Omega_k))$. Therefore using the diagonal extraction procedure, we obtain subsequences, still denoted by $\{\widehat{V}^N\}$ and $\{\widetilde{V}^N\}$ such that

$$\widehat{V}^N \to V^\dagger \quad \text{in } \mathscr{C}(0,T;H^{s-1}_{\mathrm{loc}}(\mathbb{R}^3)), \tag{3.49}$$
$$\widetilde{V}^N \to V^\ddagger \quad \text{in } \mathscr{C}(0,T;H^{s-1}_{\mathrm{loc}}(\mathbb{R}^3)). \tag{3.50}$$

Next we can check that $V^\dagger,\ V^\ddagger \in \mathscr{C}(0,T;H^{s-1}(\mathbb{R}^3))$. Given any bounded subset $\Omega \in \mathbb{R}^3$ and any $t \in [0,T]$, it follows from $\widehat{V}^N,\ \widetilde{V}^N \in \mathscr{C}(0,T;H^{s-1}(\mathbb{R}^3))$, that $\|\widehat{V}^N(t)\|_{H^{s-1}(\Omega)}$ and $\|\widetilde{V}^N(t)\|_{H^{s-1}(\Omega)}$ are bounded independently of N and from (3.49)–(3.50), we get that $\|V^\dagger(t)\|_{H^{s-1}(\Omega)}$ and $\|V^\ddagger(t)\|_{H^{s-1}(\Omega)}$ are bounded. Since this is true for any Ω we obtain $V^\dagger,\ V^\ddagger \in \mathscr{C}(0,T;H^{s-1}(\mathbb{R}^3))$.

Let us now show that $V^\dagger = V^\ddagger := V$. For each N we consider the increasing sequence $t^n = nT/N$. Therefore for each $t \in [0,T]$ we can extract a subsequence $t^{n(N)}$ such that $t^{n(N)} \to t$ when $N \to +\infty$. Consequently, we obtain in L^2_{loc}

$$\begin{aligned} V^\dagger(t) &= \lim_{N\to+\infty} \widehat{V}^N(t^{n(N)}) = \lim_{N\to+\infty} \widehat{V}_{n(N)}(t^{n(N)}) \\ &= \lim_{N\to+\infty} (\mathbb{S}_{\mathcal{T}}(t^{n(N)} - T/N, t^{n(N)}) - I)\,\widetilde{V}_{n(N)}(t^{n(N)}) + \lim_{N\to+\infty} \widetilde{V}_{n(N)}(t^{n(N)}) \\ &= \lim_{N\to+\infty} \widetilde{V}_{n(N)}(t^{n(N)}) = \lim_{N\to+\infty} \widetilde{V}^N(t^{n(N)}) = V^\ddagger(t) \end{aligned}$$

and

$$\begin{aligned} V^\ddagger(t) &= \lim_{N\to+\infty} \widetilde{V}^N(t^{n(N)}) = \lim_{N\to+\infty} \widetilde{V}_{n(N)}(t^{n(N)}) \\ &= \lim_{N\to+\infty} (\mathbb{S}_{\mathcal{L}}(t^{n(N)} - T/N, t^{n(N)}) - I)\,\widehat{V}_{n(N)-1}(t^{n(N)-1}) \\ &\quad + \lim_{N\to+\infty} \widehat{V}_{n(N)-1}(t^{n(N)-1}) \\ &= \lim_{N\to+\infty} \widehat{V}_{n(N)-1}(t^{n(N)-1}) = \lim_{N\to+\infty} \widehat{V}^N(t^{n(N)} - T/N) = V^\dagger(t), \end{aligned}$$

which proves that $V^\dagger = V^\ddagger = V$. We are now able to show that the limit point V satisfies Eq. (3.44). To this purpose we introduce the characteristic curves $X^N_\perp(t)$

associated to the transport equation (3.3)

$$\frac{d}{dt}X_\perp^N(t;t^n,x) = (\mathbb{K} \stackrel{\perp}{*} \mathcal{A}_\# \cdot \widehat{V}^N)(t, X_\perp^N(t), z), \quad t \in [t^n, t^{n+1}], \tag{3.51}$$
$$X_\perp^N(t^n;t^n,x) = x_\perp.$$

If we integrate the characteristic curves $X_\perp^N(t)$ between time t^n and t^{n+1} then we get

$$\begin{aligned}
X_\perp^N&(t^{n+1};t^n,x)\\
&= x_\perp + \int_{t^n}^{t^{n+1}} (\mathbb{K} \stackrel{\perp}{*} \mathcal{A}_\# \cdot \widehat{V}^N)(\tau, X_\perp^N(\tau;t^n,x), z)\,d\tau\\
&= x_\perp + \int_{t^n}^{t^{n+1}} \Big\{(\mathbb{K} \stackrel{\perp}{*} \mathcal{A}_\# \cdot \widetilde{V}^N)(t^{n+1}, x)\\
&\quad + \int_{\Delta t}^{t^{n+1}-s} \frac{d}{d\tau}[(\mathbb{K} \stackrel{\perp}{*} \mathcal{A}_\# \cdot \widehat{V}^N)(t^{n+1}-\tau, X_\perp^N(t^{n+1}-\tau;t^n,x), z)]\,d\tau\Big\}\,ds\\
&= x_\perp + \Delta t(\mathbb{K} \stackrel{\perp}{*} \mathcal{A}_\# \cdot \widetilde{V}^N)(t^{n+1}, x) + \mathcal{R}_1.
\end{aligned} \tag{3.52}$$

Let us show that $\mathcal{R}_1$ is bounded and scales like $\mathcal{O}(\Delta t^2)$.

$$\begin{aligned}
\mathcal{R}_1 &= \int_{t^n}^{t^{n+1}} ds \int_{\Delta t}^{t^{n+1}-s} d\tau \frac{d}{d\tau}[(\mathbb{K} \stackrel{\perp}{*} \mathcal{A}_\# \cdot \widehat{V}^N)(t^{n+1}-\tau, X_\perp^N(t^{n+1}-\tau;t^n,x), z)]\\
&= \int_{t^n}^{t^{n+1}} \int_{t^{n+1}-s}^{\Delta t} \Big\{(\mathbb{K} \stackrel{\perp}{*} \mathcal{A}_\# \cdot \partial_t \widehat{V}^N)(t^{n+1}-\tau, X_\perp^N(t^{n+1}-\tau;t^n,x), z)\\
&\quad + \sum_{\ell=1}^{2} \mathbb{K} \stackrel{\perp}{*} \mathcal{A}_\# \cdot \partial_{x_\perp^\ell} \widehat{V}^N(t^{n+1}-\tau, X_\perp^N(t^{n+1}-\tau;t^n,x), z)\\
&\quad \times (\mathbb{K}_\ell \stackrel{\perp}{*} \mathcal{A}_\# \cdot \widehat{V}^N)(t^{n+1}-\tau, X_\perp^N(t^{n+1}-\tau;t^n,x), z)\Big\}\,d\tau\,ds\\
&\le C(A_{\max}) \int_{t^n}^{t^{n+1}} ds \int_{t^{n+1}-s}^{\Delta t} d\tau \{\|\mathbb{K} \stackrel{\perp}{*} \partial_t \widehat{V}^N\|_{L^\infty([0,T]\times\mathbb{R}^3)}\\
&\quad + \|\mathbb{K} \stackrel{\perp}{*} \nabla_{x_\perp} \widehat{V}^N\|_{L^\infty([0,T]\times\mathbb{R}^3)} \|\mathbb{K} \stackrel{\perp}{*} \widehat{V}^N\|_{L^\infty([0,T]\times\mathbb{R}^3)}\}\\
&\le C(A_{\max}, \|\mathbb{K}\|_{L^1_\perp}, \|\widehat{V}^N\|_{L_t^\infty H_x^s}, \|\partial_t \widehat{V}^N\|_{L_t^\infty H_x^{s-1}})\Delta t^2.
\end{aligned} \tag{3.53}$$

Using (3.52)–(3.53), we can Taylor expand $\widehat{V}^N(t^{n+1}, X_\perp^N(t^{n+1};t^n,x), z)$ to get

$$\begin{aligned}
\widehat{V}^N&(t^{n+1}, X_\perp^N(t^{n+1};t^n,x), z)\\
&= \widehat{V}^N(t^{n+1}, x) + \Delta t(\mathbb{K} \stackrel{\perp}{*} \mathcal{A}_\# \cdot \tilde{V}^N)(t^{n+1}, x) \cdot \partial_{x_\perp} \widehat{V}^N(t^{n+1}, x)\\
&\quad + \int_{x_\perp}^{x_\perp + \Delta t(\mathbb{K} \stackrel{\perp}{*} \mathcal{A}_\# \cdot \tilde{V}^N)(t^{n+1},x) + \mathcal{R}_1} \partial_{x_\perp}^2 \widehat{V}^N(t^{n+1}, y_\perp, z)\\
&\quad \cdot (x_\perp + \Delta t(\mathbb{K} \stackrel{\perp}{*} \mathcal{A}_\# \cdot \widetilde{V}^N)(t^{n+1}, x) + \mathcal{R}_1 - y_\perp)\,dy_\perp + \mathcal{R}_1 \cdot \partial_{x_\perp} \widehat{V}^N(t^{n+1}, x)\\
&= \widehat{V}^N(t^{n+1}, x) + \Delta t(\mathbb{K} \stackrel{\perp}{*} \mathcal{A}_\# \cdot \widetilde{V}^N)(t^{n+1}, x) \cdot \partial_{x_\perp} \widehat{V}^N(t^{n+1}, x) + \mathcal{R}_2 + \mathcal{R}_3.
\end{aligned} \tag{3.54}$$

By the fact that $\widehat{V}^N(t)$ is constant along characteristic curves, and using an integration in time of Eq. (3.16) on the interval $[t^n, t^{n+1}]$ we get

$$\begin{aligned}
&\widehat{V}^N(t^{n+1}, X_\perp^N(t^{n+1}; t^n, x), z)\\
&\quad = \widetilde{V}^N(t^{n+1}, x)\\
&\quad = \widehat{V}^N(t^n, x) - \int_{t^n}^{t^{n+1}} \mathrm{Op}(\mathcal{B}(\widetilde{V}^N(t,x),\xi))\,\widetilde{V}^N(t,x)\,dt\\
&\quad = \widehat{V}^N(t^n, x) - \Delta t\,\mathrm{Op}(\mathcal{B}(\widehat{V}^N(t^n,x),\xi))\,\widehat{V}^N(t^n,x)\\
&\qquad + \int_{t^n}^{t^{n+1}}\int_{t^n}^{s}\frac{d}{d\tau}\{\mathrm{Op}(\mathcal{B}(\widetilde{V}^N(\tau,x),\xi))\,\widetilde{V}^N(\tau,x)\}\,d\tau\,ds\\
&\quad = \widehat{V}^N(t^n, x) - \Delta t\,\mathrm{Op}(\mathcal{B}(\widehat{V}^N(t^n,x),\xi))\,\widehat{V}^N(t^n,x) + \mathcal{R}_4.
\end{aligned} \tag{3.55}$$

Equating (3.54) and (3.55), multiplying the result by $\Delta t^{-1}\int_{t^n}^{t^{n+1}}\varphi(t,x)\,dt$, where $\varphi(t,x) \in \mathscr{C}_0^1([0,T]\times\mathbb{R}^3)$, integrating in space all over $\mathbb{R}^3$ and summing over n from 0 to $N-1$, we get

$$\begin{aligned}
&\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}^3}\varphi(t,x)\Bigg\{\frac{D_{\Delta t}^+\widehat{V}^N(t^n,x)}{\Delta t} + (\mathbb{K}\overset{\perp}{*}\mathcal{A}_\# \cdot \widetilde{V}^N)(t^{n+1},x)\cdot\partial_{x_\perp}\widehat{V}^N(t^{n+1},x)\\
&\qquad + \mathrm{Op}(\mathcal{B}(\widehat{V}^N(t^n,x),\xi))\,\widehat{V}^N(t^n,x)\Bigg\}\,dxdt\\
&\quad = \Delta t^{-1}\sum_{n=0}^{N-1}\sum_{i=2}^{4}\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}^3}\varphi(t,x)\mathcal{R}_i\,dxdt = \mathcal{R}_N,
\end{aligned} \tag{3.56}$$

where $D_{\Delta t}^+\widehat{V}^N(t) = \widehat{V}^N(t+\Delta t) - \widehat{V}^N(t)$. As we have

$$\left\|\frac{D_{\Delta t}^+\widehat{V}^N(t^n)}{\Delta t}\right\|_{L^\infty([0,T]\times\mathbb{R}^3)} \le \|\widehat{V}^N\|_{\mathrm{Lip}(0,T;H^{s-1}(\mathbb{R}^3))} \le C,$$

therefore we get

$$\frac{D_{\Delta t}^+\widehat{V}^N(t^n)}{\Delta t} \rightharpoonup \partial_t V, \quad \text{weakly-}* \text{ in } L^\infty(0,T;L^\infty(\mathbb{R}^3)). \tag{3.57}$$

Since we have seen that the sequences (up to extraction of subsequences) $\widehat{V}^N$ and $\widetilde{V}^N$ converge weakly in $L^\infty(0,T;H^s(\mathbb{R}^3))$ and also converge strongly in $L^\infty(0,T;L^2(\mathbb{R}^3))$ toward the limit point V, in the weakly-$*$ topology $\sigma(L^\infty_{tx}, L^1_{tx})$, we get

$$(\mathbb{K}\overset{\perp}{*}\mathcal{A}_\# \cdot \widetilde{V}^N)\cdot\partial_{x_\perp}\widehat{V}^N \rightharpoonup (\mathbb{K}\overset{\perp}{*}\mathcal{A}_\# \cdot V)\cdot\partial_{x_\perp}V, \tag{3.58}$$

$$\mathrm{Op}(\mathcal{B}(\widehat{V}^N))\,\widehat{V}^N \rightharpoonup \mathrm{Op}(\mathcal{B}(V))\,V. \tag{3.59}$$

If $\lim_{N\to\infty}\mathcal{R}_N = 0$, then using (3.57)–(3.59), Eq. (3.56) becomes in the limit

$$\int_0^T\int_{\mathbb{R}^3} \varphi(t,x)\{\partial_t V(t,x) + (\mathbb{K} \overset{\perp}{*} \mathcal{A}_{\#} \cdot V(t,x)) \cdot \partial_{x_\perp} V(t,x) + \mathrm{Op}(\mathcal{B}(V(t,x),\xi))\, V(t,x)\}\, dxdt = 0, \tag{3.60}$$

which means that the limit point V satisfies Eq. (3.44). In fact, convexity of the norm H^s implies that $\|\widehat{V}^N - V\|_{L^\infty(0,T,H^\nu(\mathbb{R}^3))} \to 0$ and $\|\widetilde{V}^N - V\|_{L^\infty(0,T,H^\nu(\mathbb{R}^3))} \to 0$ for all $\nu < s$. Since $s > n/2+1$ with $n = 3$ we can choose $\nu > n/2+1$, which shows that $V \in \mathscr{C}(0,T,\mathscr{C}^1(\mathbb{R}^3))$ is a classical solution of (3.44). In fact we can show that $V \in \mathscr{C}(0,T;H^s(\mathbb{R}^3)) \cap \mathscr{C}^1(0,T;H^{s-1}(\mathbb{R}^3))$.

Let us now show that $\lim_{N\to\infty}\mathcal{R}_N = 0$. Let us begin with $\mathcal{R}_3$. Using estimate (3.53) we then have

$$\begin{aligned}
&\Delta t^{-1}\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}^3}\varphi(t,x)\mathcal{R}_3\, dxdt \\
&\quad \le \Delta t^{-1}\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}^3}|\varphi(t,x)|\|\mathcal{R}_1\partial_{x_\perp}\widehat{V}^N\|_{L^\infty([0,T]\times\mathbb{R}^3)}\, dxdt \\
&\quad \le C(A_{\max}, \|\mathbb{K}\|_{L^1_\perp}, \|\widehat{V}^N\|_{L^\infty_t H^s_x}, \|\partial_t\widehat{V}^N\|_{L^\infty_t H^{s-1}_x}, \|\varphi\|_{L^1_{tx}})\Delta t.
\end{aligned} \tag{3.61}$$

Let us now deal with $\mathcal{R}_4$. Using Eq. (3.16) we get

$$\begin{aligned}
&\Delta t^{-1}\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}^3}\varphi(t,x)\mathcal{R}_4\, dtdx \\
&\quad = \Delta t^{-1}\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}^3}\int_{t^n}^{t^{n+1}}\int_{t^n}^{s}\bigg\{\varphi(t,x)\frac{d}{d\tau}\mathrm{Op}(\mathcal{B}(\widetilde{V}^N(\tau,x),\xi))\,\widetilde{V}^N(\tau,x) \\
&\qquad - \varphi(t,x)\mathrm{Op}(\mathcal{B}(\widetilde{V}^N(\tau,x),\xi))^2\,\widetilde{V}^N(\tau,x)\bigg\}\, d\tau\, dsdtdx = \mathcal{R}_{41} + \mathcal{R}_{42}.
\end{aligned} \tag{3.62}$$

The first term $\mathcal{R}_{41}$ of (3.62) can be estimated as follows:

$$\begin{aligned}
\mathcal{R}_{41} &\le \Delta t^{-1}\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}} dt\int_{\mathbb{R}^3} dx|\varphi(t,x)| \\
&\quad \cdot \int_{t^n}^{t^{n+1}} ds\int_{t^n}^{s} d\tau\left\|\frac{d}{d\tau}\mathrm{Op}(\mathcal{B}(\widetilde{V}^N))\,\widetilde{V}^N\right\|_{L^\infty([0,T]\times\mathbb{R}^3)} \\
&\le C(A_{\max})\|\partial_z\widetilde{V}^N\|_{L^\infty([0,T]\times\mathbb{R}^3)}\|\partial_t\widetilde{V}^N\|_{L^\infty([0,T]\times\mathbb{R}^3)}\|\varphi\|_{L^1([0,T]\times\mathbb{R}^3)}\Delta t \\
&\le C(A_{\max}, \|\widetilde{V}^N\|_{L^\infty_t H^s_x}, \|\partial_t\widetilde{V}^N\|_{L^\infty_t H^{s-1}_x}, \|\varphi\|_{L^1_{tx}})\Delta t.
\end{aligned} \tag{3.63}$$

The second term $\mathcal{R}_{42}$ of (3.62) can be bounded as follows:

$$
\begin{aligned}
\mathcal{R}_{42} &\leq \Delta t^{-1} \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \int_{t^n}^{t^{n+1}} \int_{t^n}^{s} |\langle \mathrm{Op}(\mathcal{B}(\widetilde{V}^N(\tau)))^2 \widetilde{V}^N(\tau), \varphi(t)\rangle| d\tau ds dt \\
&\leq \Delta t^{-1} \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} dt \int_{t^n}^{t^{n+1}} ds \\
&\quad \cdot \int_{t^n}^{s} d\tau |\langle \mathrm{Op}(\mathcal{B}(\widetilde{V}^N(\tau))) \widetilde{V}^N(\tau), \mathrm{Op}(\mathcal{B}(\widetilde{V}^N(\tau)))^\star \varphi(t)\rangle| \\
&\leq T\Delta t \|\mathrm{Op}(\mathcal{B}(\widetilde{V}^N)) \widetilde{V}^N\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \|\mathrm{Op}(\mathcal{B}(\widetilde{V}^N))^\star \varphi\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \\
&\leq T\Delta t \|\mathrm{Op}(\widetilde{q}(\widetilde{V}^N))\partial_z \widetilde{V}^N\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \{\|\mathrm{Op}(q^\star(\widetilde{V}^N))\varphi\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \\
&\quad + \|(\mathrm{Op}(q(\widetilde{V}^N))^\star - \mathrm{Op}(q^\star(\widetilde{V}^N)))\varphi\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\} \\
&\leq C\Delta t \|\widetilde{V}^N\|_{L^\infty([0,T]\times\mathbb{R}^3)} \|\partial_z \widetilde{V}^N\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \\
&\quad \cdot \{\|\widetilde{V}^N\|_{L^\infty([0,T]\times\mathbb{R}^3)} \|\partial_z \varphi\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \\
&\quad + \|\widetilde{V}^N\|_{L^\infty(0,T;\mathscr{C}^1(\mathbb{R}^3))} \|\varphi\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\} \\
&\leq C(A_{\max}, \|\widetilde{V}^N\|_{L_t^\infty H_x^s}, \|\varphi\|_{L_t^\infty H_x^1}) \Delta t.
\end{aligned} \tag{3.64}
$$

Let us show that error term associated to $\mathcal{R}_2$ is bounded and scales like $\mathcal{O}(\Delta t)$. Using estimate (3.53) we have

$$
\begin{aligned}
\Delta \eta^N(x) &= \Delta t (\mathbb{K} \overset{\perp}{*} \mathcal{A}_\# \cdot \widetilde{V}^N)(t^{n+1}, x) + \mathcal{R}_1 \\
&\leq C(A_{\max}, \|\mathbb{K}\|_{L^1_\perp}, \|\widehat{V}^N\|_{L_t^\infty H_x^s}, \|\widetilde{V}^N\|_{L_t^\infty H_x^{s-1}}, \|\partial_t \widehat{V}^N\|_{L_t^\infty H_x^{s-1}}) \Delta t \\
&\leq C_\natural \Delta t.
\end{aligned} \tag{3.65}
$$

Using (3.65) we deduce

$$
\begin{aligned}
&\Delta t^{-1} \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}^3} \varphi(t^n, x) \mathcal{R}_2 dx dt \\
&\quad = \Delta t^{-1} \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} dt \int_{\mathbb{R}^3} dx \varphi(t, x) \\
&\qquad \cdot \int_0^{\Delta \eta^N(x)} y_\perp \partial_{x_\perp}^2 \widehat{V}^N \big(t^{n+1}, x_\perp + \Delta \eta^N(x) - y_\perp, z\big) dy_\perp \\
&\quad \leq \Delta t^{-1} \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} dt \int_{\mathbb{R}^3} dx |\varphi(t, x)| \\
&\qquad \cdot \left(\int_0^{C_\natural \Delta t} |y_\perp|^2 dy_\perp \right)^{1/2} \left(\int_{\mathbb{R}^2} |\partial_{x_\perp}^2 \widehat{V}^N(t^{n+1}, y_\perp, z)|^2 dy_\perp \right)^{1/2} \\
&\quad \leq \sqrt{\frac{\pi}{2}} C_\natural^2 T \Delta t \|\varphi\|_{L^\infty(0,T;L^2(\mathbb{R}_z;L^1(\mathbb{R}_\perp^2)))} \|\partial_{x_\perp}^2 \widehat{V}^N\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \\
&\quad \leq C(A_{\max}, \|\mathbb{K}\|_{L^1_\perp}, \|\widehat{V}^N\|_{L_t^\infty H_x^s}, \|\widetilde{V}^N\|_{L_t^\infty H_x^{s-1}}, \|\partial_t \widehat{V}^N\|_{L_t^\infty H_x^{s-1}}, \|\varphi\|_{L_t^\infty L_z^2 L_\perp^1}) \Delta t.
\end{aligned} \tag{3.66}
$$

Finally using *a priori* estimates (3.61)−(3.64) and (3.66), we get

$$\mathcal{R}_N = \mathcal{O}(\Delta t) \quad \text{and thus} \quad \lim_{N\to\infty} \mathcal{R}_N = 0,$$

which ends the proof. □

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