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Discontinuous Galerkin finite element methods for the gyrokinetic-waterbag equations

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In this article, we design and analyse discontinuous Galerkin methods for approximating the gyrokinetic-waterbag equations in a cylindrical geometry. This model, which attempts to describe turbulent flows in magnetically confined fusion plasmas, results from special but exact weak solutions of the gyrokinetic-Vlasov equations. Actually, using geometric Liouville invariants, the waterbag reduction concept reduces exactly Vlasov kinetic equations into multi-fluid systems of conservation laws, with nonlocal fluxes, for level lines (called contours) of the phase space. The schemes are constructed by coupling a discontinuous Galerkin approximation, with different numerical fluxes, for the contour equations representing the particle distribution of the plasma, together with a discontinuous or a continuous Galerkin finite element method for the quasi-neutrality equation giving self-consistent electrical potential from the particle distribution. In the case of smooth compactly supported solutions, we show high-order error estimates in the L^2-norm of order $\min(k, k')$ for the approximation of contours and of order $\min(k, k') + 1$ for the electrical potential approximation, where $k \geq 5/2$ (respectively $k' \geq 5/2$) is the degree of polynomial reconstructions for contours (respectively electrical potential). We also prove that the total energy of the approximate solution is nonincreasing for the drift-kinetic case and asymptotically nonincreasing for the gyrokinetic case.

Keywords: gyrokinetic-waterbag model; gyrowaterbag model; discontinuous Galerkin methods; finite element methods; gyrokinetic-Vlasov equations; multi-fluid systems; infinite-dimensional hyperbolic system of conservation laws in several space dimensions; magnetically confined fusion plasmas.

1. Introduction

The problem of turbulence in magnetically confined fusion plasmas is a major issue to achieve self-sustained fusion reactions. Indeed heat, particle and momentum transport, which are crucial for plasma confinement in fusion devices, usually result from turbulent processes. Such turbulent flows in three-dimensional cylindrical geometry require a kinetic description given by the gyrokinetic-Vlasov equations (2.1) and (2.2) derived in Dubin *et al.* (1983) for example.

Although more accurate, the gyrokinetic Vlasovian description of turbulent transport is much more demanding in computer resources than fluid simulations, which are less accurate and sometimes very far from experimental measurements. The main reason is that fluid modelling does not take into account wave–particle interaction, such as Landau damping, which plays an important role in gyrokinetic turbulence. This motivated us to use an alternative approach based on the waterbag reduction concept detailed in Besse *et al.* (2009). Actually, the waterbag reduction concept uses the geometric Liouville invariants, i.e., preservation of phase-space volume, to design special but exact weak solutions, which are called waterbag distribution functions and are constructed as a sum of characteristic functions (2.5). It turns out that the waterbag distribution function reduces exactly (without ad hoc closure) the gyrokinetic-Vlasov equations (2.1) and (2.2) into a multi-fluid system of conservation laws with nonlocal fluxes that we call the gyrokinetic-waterbag equations (2.6) and (2.7). Therefore, this model creates the bridge between fluid and kinetic descriptions of a collisionless plasma, allowing us to keep the kinetic aspect of the problem (such as Landau damping and resonant wave–particle interaction) with the same complexity as a multi-fluid model in three-dimensional space (Besse *et al.*, 2009).

The main feature of the gyrokinetic-waterbag equations is to be incompressible in the two-dimensional transverse direction and compressible in the one-dimensional longitudinal direction. Numerical simulations based on semi-Lagrangian (SL) schemes and a cubic-spline reconstruction have given convincing and good results (Besse *et al.*, 2008; Besse & Bertrand, 2009; Coulette & Besse, 2013a,b). Even if SL schemes give satisfactory numerical results, they present some drawbacks. The first one concerns the nature of the solution. Indeed SL schemes rely on the integration of characteristic curves whose existence is ensured only in the case of classical smooth solutions but fails when we deal with weak solutions. Actually, the gyrokinetic-waterbag equations can justly develop weak solutions in large time, namely shocks, because of the presence of the compressible Burgers-like term in the parallel direction (see (2.6)). Therefore, to follow the long-time behaviour of the solution, we need a numerical scheme which is able to capture this weak solution. For this purpose, SL schemes are useless and may be wrong. The second drawback of an SL method is its difficult and poor parallelization because this scheme requires us to compute the origin of a set of characteristic curves, which end at the mesh vertices and change from time to time. Finally, the third and the fourth drawbacks are, respectively, *hp*-refinement and treatment of complex geometry with complicated boundary conditions. These four drawbacks are solved when one uses a discontinuous Galerkin method. Indeed they are known to be highly parallelizable, stable, high-order accurate, well adapted to *hp*-refinement and designed to capture weak solution of hyperbolic systems of conservation laws (Cockburn & Shu, 2001) in complex geometries and with complicated boundary conditions. Moreover, recent applications of discontinuous Galerkin methods to fluid and kinetic models for plasma physics (Liu & Shu, 2000; Liu & Xin, 2000; Besse *et al.*, 2008, 2009; Besse & Bertrand, 2009; Ayuso de Dios *et al.*, 2011, 2012; Cheng *et al.*, 2013, 2014b,c,a; Madaule *et al.*, 2014; Liu & Ploymaklam, 2015) have us even more motivated to consider this method for the gyrokinetic-waterbag equations (2.6) and (2.7).

Since the gyrokinetic-waterbag equations (2.6) and (2.7) can be viewed as an infinite-dimensional system, the first step is to convert it into a finite-dimensional system (2.8)–(2.9) by considering a finite number of bags. Here, a bag denotes the geometric area delimited by the contours of a same level line of the phase space. Bags can be interpreted as different independent fluids (see Section 2.2). The schemes are constructed by coupling a discontinuous Galerkin approximation with different numerical fluxes for contour equations (representing the particle distribution of the plasma) together with a discontinuous or a continuous Galerkin finite element method for the quasi-neutrality equation (giving the self-consistent electrical potential from the particle distribution). In the case of smooth compactly supported solutions, we show high-order error estimates in the L^2-norm of order $\min(k, k')$ for the approximation of contours and of order $\min(k, k') + 1$ for the electrical potential approximation. Here, $k \geq 5/2$ (respectively $k' \geq 5/2$) is the degree of polynomial reconstructions for contours (respectively electrical potential). The order of convergence $\min(k, k')$ for contour approximations is suboptimal with respect to the order $\min(k, k') + 1$ expected when we use a polynomial reconstruction of degree k (for contours) and k' (for the electrical potential). This restriction of convergence order comes from the source term played by the parallel derivative of the electrical potential in the contour equations (2.8). For more explanation of about this limitation and some ideas to recover optimal order of convergence, we refer to Remarks 4.2, 4.7, 4.8, 4.10, 4.12.

1.1 *Organization of the article*

In Section 2, we present the continuous problem, by describing briefly, first the gyrokinetic-Vlasov model (2.6)–(2.7) in Section 2.1, second the transition to a finite-dimensional gyrokinetic-waterbag system (2.8)–(2.9) in Section 2.2 and finally by giving an existence, uniqueness and regularity result for classical local-in-time solutions of the system (2.8)–(2.9). Section 3 exposes the discrete problem built from discontinuous Galerkin methods and useful tools for its analysis. Section 4 starts with the main theorem (Theorem 4.1) of this article, which states the convergence and error estimates of the numerical scheme. This section carries on with the proof of Theorem 4.1. Eventually, it ends with the proof of some stability properties of the scheme in L^2- and L^∞-norms, which are useful for estimating total energy. In Section 5, we prove that the total energy of the approximate solution is nonincreasing for the drift-kinetic case and asymptotically nonincreasing for the gyrokinetic case. Finally, Appendix A contains the proofs of error estimates for the electrical potential approximation with respect to those for the contours, in L^2- and L^∞-norms. The latter is needed to deal with the nonlinearity of the advection field.

2. The continuous problem

In this section, we present the continuous model, namely the gyrokinetic-waterbag equations (see Section 2.2), which results from special but exact weak solutions of the gyrokinetic-Vlasov equations (see Section 2.1). These weak solutions make use of the 'waterbag invariants', expressing, on the one hand, the conservation of the distribution function along phase-space characteristics and, on the other hand, the conservation of phase-space volume (Liouville's theorem).

2.1 *The gyrokinetic-Vlasov model*

Three-dimensional space is decomposed into a one-dimensional longitudinal or parallel direction e_3, denoted by the symbol $\|$, which is the direction parallel to the magnetic field $B = B_0 b$ (e.g., parallel to the axis of a cylindrical column, i.e., $b = e_3$), and a two-dimensional transverse or perpendicular direction, denoted by the symbol $\perp$, which is perpendicular to b. The variable associated to the parallel direction is denoted by $x_\| = x_3 \in \mathbb{R}$, while the one associated to the perpendicular direction is denoted by $x_\perp = (x_1, x_2) \in \mathbb{R}^2$. Finally, we introduce the one-dimensional independent velocity variable $v_\| \in \mathbb{R}$, which describes the acceleration of particles induced by the electrical potential ϕ. Moreover, we use the notation $\nabla = (\partial_1, \partial_2, \partial_3)^{\mathrm{T}} = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})^{\mathrm{T}} = (\nabla_{x_\perp}^{\mathrm{T}}, \partial_{x_\|})^{\mathrm{T}} = (\nabla_\perp^{\mathrm{T}}, \partial_\|)^{\mathrm{T}}$.

Within gyrokinetic Hamiltonian formalism and cylindrical geometry framework (Dubin *et al.*, 1983), the gyrokinetic-Vlasov equation expresses the fact that the ion gyrocentre distribution function $f = f(t, x, v_\|, \mu) = f(t, x_1, x_2, x_3, v_\|, \mu)$ is constant along gyrocentre characteristic curves in gyrocentre phase space $(t, x, v_\|, \mu) \in \mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_{v_\|} \times \mathfrak{M}$:

$$\partial_t f + \mathcal{J}_\perp v_E \cdot \nabla_\perp f + v_\| \partial_\| f + \frac{q_i}{m_i} \mathcal{J}_\perp E_\| \partial_{v_\|} f = 0 \quad \forall \mu \in \mathfrak{M}. \tag{2.1}$$

The ion distribution function f is coupled to the electrical potential ϕ via the quasi-neutrality equation

$$-\nabla_\perp \cdot \left(\frac{n_{i0}}{B_0 \Omega_0} \nabla_\perp \phi \right) + \frac{e \tau n_{i0}}{k_B T_{i0}} (\phi - \lambda \langle \phi \rangle_\|) = \int_{\mathfrak{M}} \int_{\mathbb{R}} \mathcal{J}_\perp f(t, x, v_\|, \mu) \, \mathrm{d}v_\| \, \mathrm{d}\mu - n_{i0}. \tag{2.2}$$

The term $\langle\phi\rangle_{\parallel}$ denotes the average of the electrical potential ϕ over a magnetic field line, which is a straight line parallel to the direction e_3. Let us remark that differential equation (2.2) is elliptic only in the transverse direction.

In (2.1) and (2.2), $q_i = Z_i e$ and m_i are, respectively, the ion charge and mass; $Z_i n_{i0} = n_{e0}(x_\perp)$ is the electronic density; $T_{e0} = T_{e0}(x_\perp)$ is the electronic temperature; $\tau = T_{i0}/T_{e0}$, $\lambda \in \{0, 1\}$; $E = -\nabla\phi$ is the electric field; $E_\parallel = E \cdot b$ and $v_E = E \times B/B_0^2 = \nabla_\perp^\perp \phi/B_0 = (-\partial_{x_2}\phi, \partial_{x_1}\phi)^{\mathrm{T}}/B_0$ is the electrical drift velocity. Let us note that the magnetic moment μ is an invariant and thus it must be considered a parameter or a label but not a differential variable. The invariant μ belongs to an open subset $\mathfrak{M}$ of $\mathbb{R}^+$, while $\mathrm{d}\mu$ stands for the Lebesgue measure. In addition to cylindrical geometry, we have supposed that the magnetic field B is uniform and constant along the axis of the column (i.e., $B_0 =$ constant and b a constant unit vector), which means that the perpendicular (with respect to the unit vector $b = e_3$) drift velocity does not admit any magnetic curvature or gradient effect and that the ion cyclotron frequency, $\Omega_0 = q_i B_0/m_i$, is a constant.

Finally, the integral operator $\mathcal{J}_\perp$ stands for the gyroaverage operator defined by

$$\mathcal{J}_\perp f(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\zeta f(x + \rho(\zeta)), \tag{2.3}$$

where ζ is the gyroangle. The gyroradius vector ρ is given by $\rho(\zeta) = \sqrt{2\mu/(q_i\Omega_0)}\hat{a}(\zeta)$, where the vector $\hat{a}(\zeta) = \hat{x}\cos\zeta - \hat{y}\sin\zeta$ is defined in terms of the fixed local unit vector basis $(\hat{x}, \hat{y}, b = \hat{x} \times \hat{y})$. Using Fourier transforms, the gyroaverage operator reads

$$\mathcal{J}_\perp f(x) = \int_{\mathbb{R}^3} \mathrm{d}k \hat{f}(k) \exp(\mathrm{i}k \cdot x) J_0\left(k_\perp \sqrt{\frac{2\mu}{q_i\Omega_0}}\right), \tag{2.4}$$

where $k_\perp^2 := k_{\hat{x}}^2 + k_{\hat{y}}^2$, and J_0 is the Bessel function of the first kind and order zero.

Since the magnetic moment μ is not an independent variable but a parameter or a label related to an invariant, we can consider the plasma as a superposition of a (possibly uncountable) collection of bunches of particles having the same initial magnetic moment μ. This standard approach is equivalent mathematically to considering solutions of the Vlasov equation (2.1) written as

$$f(t, x, v_\parallel, \mu) = \int_{\mathfrak{M}} f_\nu(t, x, v_\parallel)\delta_\nu(\mu)\mathfrak{m}(\mathrm{d}\nu).$$

Here $\delta_\nu(\mu)$ is the Dirac mass, ν is a parameter belonging to some probability space $\mathfrak{M}$, $\mathfrak{m}$ is a probability measure on that space and f_ν are smooth functions, which still satisfy the Vlasov equation (2.1) with $\mu = \nu$. For instance we could take the discrete measure $\mathfrak{m}(\mathrm{d}\nu) = \sum_l \varpi_\ell \delta(\nu - \mu_l)$, where ϖ_ℓ are positive constants. As a consequence the distribution function f can be recast as

$$f(t, x, v_\parallel, \mu) = \sum_\ell \varpi_\ell f_{\mu_\ell}(t, x, v_\parallel)\delta(\mu - \mu_\ell),$$

where the function $f_{\mu_\ell}(t, x, v_\parallel)$ satisfies the Vlasov equation (2.1) (with $\mu = \mu_\ell$) for all values of the index ℓ.

2.2 *The gyrokinetic-waterbag model*

For the description of the waterbag reduction concept applied to one-dimensional Vlasov equations we refer the reader to Besse *et al.* (2009). We now apply it to the gyrokinetic-Vlasov equation (2.1). For this purpose, for every magnetic moment $\mu \in \mathfrak{M}$, we consider two three-dimensional Lagrangian foliations, denoted $v_\mu^\pm(t,x,a)$, of codimension one, of the four-dimensional phase space $(x, v_\parallel) \subset \mathbb{R}^4$, which may be viewed as families of three-dimensional smooth functions $v_\mu^\pm(t,x,a)$, labeled by the one-dimensional index a, belonging to the set $\mathcal{A} = [0,1]$. The collection $\{v_\mu^j\}_{\mu\in\mathfrak{M}}$, with $j \in \{-,+\}$ is represented in compact form by $v^j = v^j(t,x,a,\mu)$. Therefore, the leaves $v^\pm$ can be reinterpreted as two three-dimensional Lagrangian foliations of codimension two, of the five-dimensional phase space $(x, v_\parallel, \mu) \subset \mathbb{R}^3\times\mathbb{R}\times\mathbb{R}^+$, where leaves are labeled by the couple $\sigma = (a,\mu) \in \mathfrak{L} = \mathcal{A}\times\mathfrak{M}$. We suppose that the leaves $v^\pm$ are smooth functions such that $v^- \le v^+$, $\partial_a v^+ \le 0$, $\partial_a v^- \ge 0$. Considering two nonclosed single-valued smooth branches $v^\pm(t,x,a,\mu)$ of the $(x, v_\parallel, \mu)$-phase space, we can define $f_\mu(t,x,v_\parallel,\mu)$ such that

$$f_\mu(t,x,v_\parallel,\mu) = \int_{\mathcal{A}} \left(\mathcal{H}\left(v^+(t,x,a,\mu) - v_\parallel\right) - \mathcal{H}\left(v^-(t,x,a,\mu) - v_\parallel\right)\right) m(\mathrm{d}a), \tag{2.5}$$

where m denotes a probability measure on $\mathcal{A}$ and $\mathcal{H}$ is the Heaviside unit step function. In the distribution function (2.5), choosing $m(\mathrm{d}a) = \sum_i \mathcal{A}_i\delta(a-a_i)$ (where $\mathcal{A}_i$ are positive constants), we then recover the multiple waterbag distribution (Besse *et al.*, 2009).

As long as the contours $v^\pm$ are smooth, single valued and do not cross, the waterbag distribution function (2.5) is an exact weak solution of the gyrokinetic-Vlasov equation (2.1) in the sense of distribution theory, if and only if the following set of equations is satisfied,

$$\partial_t v^\pm + \nabla_\perp\cdot(\mathcal{J}_\perp v_E v^\pm) + \partial_\parallel\left(\frac{1}{2}v^{\pm 2} + \frac{q_i}{m_i}\mathcal{J}_\perp\phi\right) = 0 \quad \forall a\in\mathcal{A},\ \forall\mu\in\mathfrak{M}, \tag{2.6}$$

and are coupled to the quasi-neutrality equation taking the form

$$-\nabla_\perp\cdot\left(\frac{n_{i0}}{B_0\Omega_0}\nabla_\perp\phi\right) + \frac{e\tau n_{i0}}{k_B T_{i0}}(\phi - \lambda\langle\phi\rangle_\parallel) = \int_{\mathcal{A}} m(\mathrm{d}a)\int_{\mathfrak{M}} \mathfrak{m}(\mathrm{d}\mu)\,\mathcal{J}_\perp(v^+ - v^-) - n_{i0}. \tag{2.7}$$

Since we are looking for a numerical approximation of the system (2.6)–(2.7), i.e., with unknowns living in a finite-dimensional space, we choose for the probability measures m and $\mathfrak{m}$, the following finite counting measures:

$$m(\mathrm{d}a) = \sum_{i=1}^{\mathrm{N}} \mathcal{A}_i\delta(a-a_i) \quad \text{and} \quad \mathfrak{m}(\mathrm{d}\nu) = \sum_{\ell=1}^{\mathrm{M}} \varpi_\ell\delta(\nu-\mu_\ell).$$

Here, $\mathcal{A}^{\mathrm{N}} = \{a_i\}_{i=1,\dots,\mathrm{N}}$ (respectively $\mathfrak{M}^{\mathrm{M}} = \{\mu_\ell\}_{\ell=1,\dots,\mathrm{M}}$) is a discrete sequence of numbers belonging to the set $\mathcal{A}$ (respectively $\mathfrak{M}$) and $\{\mathcal{A}_i\}_{i=1,\dots,\mathrm{N}}$ (respectively $\{\varpi_\ell\}_{\ell=1\dots\mathrm{M}}$) are positive real numbers with N (respectively M) a positive integer. Introducing the positive numbers $\mathcal{A}_\sigma \equiv \mathcal{A}_{a\mu} := \mathcal{A}_i\varpi_\ell$, where for each couple $\sigma = (a,\mu) \in \mathfrak{L}^{\mathrm{NM}} := \mathcal{A}^{\mathrm{N}}\times\mathfrak{M}^{\mathrm{M}}$ there corresponds a unique couple $(i,\ell) \in [1,\mathrm{N}]\times[1,\mathrm{M}]$, and considering the unknown contours $\{v_\sigma^\pm\}_{\sigma\in\mathfrak{L}^{\mathrm{NM}}}$, the infinite-dimensional system (2.6)–(2.7) is then recast as the finite-dimensional system

$$\partial_t v_\sigma^\pm + \nabla_\perp\cdot(\mathcal{J}_\perp v_E v_\sigma^\pm) + \partial_\parallel\left(\frac{1}{2}v_\sigma^{\pm 2} + \frac{q_i}{m_i}\mathcal{J}_\perp\phi\right) = 0 \quad \forall\sigma\in\mathfrak{L}^{\mathrm{NM}}, \tag{2.8}$$

and the quasi-neutrality equation takes the form

$$-\nabla_\perp \cdot \left(\frac{n_{i0}}{B_0 \Omega_0} \nabla_\perp \phi \right) + \frac{e\tau n_{i0}}{T_{i0}} (\phi - \lambda \langle \phi \rangle_\parallel) = \sum_{\sigma \in \mathfrak{L}^{\mathrm{NM}}} \mathcal{A}_\sigma \, \mathcal{J}_\perp (v_\sigma^+ - v_\sigma^-) - n_{i0}. \tag{2.9}$$

In the previous equations, let us recall that the operator $\mathcal{J}_\perp$, defined by (2.4), depends parametrically on $\mu \in \mathfrak{M}^{\mathrm{M}}$ and thus on $\sigma \in \mathfrak{L}^{\mathrm{NM}}$. Moreover, we add to the systems (2.8)–(2.9) initial conditions $v_\sigma^\pm(0, \cdot) = v_{0\sigma}^\pm(\cdot)\ \forall \sigma \in \mathfrak{L}^{\mathrm{NM}}$.

From Besse (2011); Bardos & Besse (2013, 2015) we obtain the following well-posedness result for local-in-time classical solutions of the gyrowaterbag equations (2.8) and (2.9).

THEOREM 2.1 We assume that the given positive functions $x_\perp \mapsto n_{i0}$ and $x_\perp \mapsto T_{i0}$ are regular enough and that initial conditions $v_{0\sigma}^\pm \in H^s(\mathbb{R}^3)\ \forall \sigma \in \mathfrak{L}^{\mathrm{NM}}$, with $s > 5/2$. We suppose that for every $\mu \in \mathfrak{M}^{\mathrm{M}}$, the map $a \mapsto v_{0a\mu}^-$ (respectively $a \mapsto v_{0a\mu}^+$) is nondecreasing (respectively nonincreasing) on the set $\mathcal{A}^{\mathrm{N}}$ and that there exist positive constants $m_{0\mu}$ and $M_{0\mu}$ such that $m_{0\mu} < \pm v_{0a\mu}^\pm < M_{0\mu}\ \forall a \in \mathcal{A}^{\mathrm{N}}$. Then there exists a time T, such that

$$\begin{cases} v_\sigma^\pm \in L^\infty(0, T;\, H^s(\mathbb{R}^3)) \cap \mathrm{Lip}(0, T;\, H^{s-1}(\mathbb{R}^3)) \quad \forall \sigma \in \mathfrak{L}^{\mathrm{NM}}, \\ \partial_\perp^\alpha \phi \in L^\infty(0, T;\, H^s(\mathbb{R}^3)) \cap \mathrm{Lip}(0, T;\, H^{s-1}(\mathbb{R}^3)), \quad |\alpha| \le 2. \end{cases}$$

Moreover, there exist positive constants m_μ and M_μ, such that for all time $t \in [0, T)$ and for every $\mu \in \mathfrak{M}^{\mathrm{M}}$, the map $a \mapsto v_{a\mu}^-$ (respectively $a \mapsto v_{a\mu}^+$) is nondecreasing (respectively nonincreasing) on the set $\mathcal{A}^{\mathrm{N}}$ and $v_{a\mu}^\pm$ satisfy

$$m_\mu < \pm v_{a\mu}^\pm < M_\mu \quad \forall a \in \mathcal{A}^{\mathrm{N}}.$$

Finally, we end this section by giving the mass and energy conservation laws satisfied by (2.8) and (2.9). Let $\Omega \subset \mathbb{R}^3$ be the support of contours, mass conservation follows (after a sum over $\sigma \in \mathfrak{L}^{\mathrm{NM}}$) from the relations

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathcal{A}_\sigma (v_\sigma^+ - v_\sigma^-) \mathrm{d}x = 0 \quad \forall \sigma \in \mathfrak{L}^{\mathrm{NM}},$$

which express Liouville geometric invariants preservation. Combining (2.8) and (2.9), we obtain the following energy conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{\sigma \in \mathfrak{L}^{\mathrm{NM}}} \frac{1}{6} \int_\Omega \mathcal{A}_\sigma \left((v_\sigma^+)^3 - (v_\sigma^-)^3 \right) \mathrm{d}x + \frac{1}{2} \frac{q_i}{m_i} \int_\Omega \mathrm{d}x \left(\frac{n_{i0}}{B_0 \Omega_0} |\nabla_\perp \phi|^2 + \frac{e\tau n_{i0}}{k_B T_{i0}} |\phi - \langle \phi \rangle_\parallel|^2 \right) \right) = 0. \tag{2.10}$$

3. The discrete problem

In this section, we present a discretization of the problem (2.8)–(2.9) by using discontinuous Galerkin finite element schemes (see Section 3.2). Before giving the numerical method, we start by defining some tools (see Section 3.1), which will be useful to present the scheme and perform its analysis (see Section 4).

3.1 *Preliminaries and notation*

In this section, we fix some notation and give the definition of useful tools to state the numerical scheme (see Section 3.2) and perform its analysis (see Section 4). First, we set some definitions related to the domain and boundary conditions (see Section 3.1.1). Second, we introduce the spaces of discretization (see Section 3.1.2). Finally, we expose the properties of the projection-interpolation (see Section 3.1.3) and gyroaverage (see Section 3.1.4) operators, which are needed to show the convergence and high-order error estimates of the scheme.

3.1.1 *Domain definitions and boundary conditions.* Let $\Omega = \Omega_\perp \times \Omega_\parallel$ be a bounded convex polyhedral domain of $\mathbb{R}^3$ constructed as the Cartesian product of a bounded convex polygonal domain $\Omega_\perp$ of $\mathbb{R}^2$ and a bounded domain $\Omega_\parallel$ of $\mathbb{R}$. Therefore, the boundary $\Gamma_\perp$ (respectively $\Gamma_\parallel$) of Ω in the perpendicular (respectively parallel) direction is given by $\Gamma_\perp = \partial\Omega_\perp \times \Omega_\parallel$ (respectively $\Gamma_\parallel = \Omega_\perp \times \partial\Omega_\parallel$). Let ϕ_b and $v^\pm_{b\sigma}$, $\forall \sigma \in \mathfrak{L}^{\mathrm{NM}}$, be given regular functions defined on the boundary $\partial\Omega_\perp$ and $\tau_\perp$ the unit tangential vector to $\partial\Omega_\perp$ defined by $\tau_\perp = \nu_\perp^\perp$. The vector $\nu_\perp^\perp$ is the counterclockwise $\pi/2$-rotation of the unit outward normal vector $\nu_\perp$ to the boundary $\partial\Omega_\perp$. We can then consider three types of boundary conditions:

$$
\begin{aligned}
\text{Dirichlet}: & \ \phi_{|\Gamma_\perp} = \phi_b \ \text{ in the } \perp\text{-direction and } \phi \text{ is } L_\parallel\text{-periodic in the } \parallel\text{-direction;}\\
& \ \forall \sigma \in \mathfrak{L}^{\mathrm{NM}},\ v^\pm_{\sigma\,|\Gamma_\perp} = v^\pm_{b\sigma} \ \text{ in the } \perp\text{-direction and } v^\pm_\sigma \text{ is } L_\parallel\text{-periodic in the } \parallel\text{-direction.}\\
\text{Neumann}: & \ \nabla\phi_{|\Gamma_\perp} \cdot \tau_\perp = \phi_b \ \text{ in the } \perp\text{-direction and } \phi \text{ is } L_\parallel\text{-periodic in the } \parallel\text{-direction;}\\
& \ \forall \sigma \in \mathfrak{L}^{\mathrm{NM}},\ v^\pm_{\sigma\,|\Gamma_\perp} = v^\pm_{b\sigma} \ \text{ in the } \perp\text{-direction and } v^\pm_\sigma \text{ is } L_\parallel\text{-periodic in the } \parallel\text{-direction.}\\
\text{Periodic}: & \ \phi \textit{ is} \text{ periodic in all directions and } \forall \sigma \in \mathfrak{L}^{\mathrm{NM}},\ v^\pm_{\sigma\,|\Gamma_\perp} \ \text{ periodic in all directions.}\\
& \ \text{In this case } \Omega \text{ is the periodic box } \Omega = \mathbb{T}^3_L = \prod_{i=1}^{3} (\mathbb{R}/L_i\mathbb{Z}), \ \text{ with } L_i > 0,\ 1 \le i \le 3.
\end{aligned}
$$

3.1.2 *Spaces of discretization*

Domain partitioning and trace operators. Let $\mathcal{T}_{h_\perp}$ (respectively $\mathcal{I}_{h_\parallel}$) be the family of partitions of $\Omega_\perp$ (respectively $\Omega_\parallel$) constituted of rectangles or triangles (respectively segments). The family of partitions $\mathcal{M}_h$ of Ω is constructed as the Cartesian product of the partitions $\mathcal{T}_{h_\perp}$ and $\mathcal{I}_{h_\parallel}$, i.e.,

$$\mathcal{M}_h = \mathcal{T}_{h_\perp} \times \mathcal{I}_{h_\parallel} := \left\{ K = K_\perp \times K_\parallel,\ \ K_\perp \in \mathcal{T}_{h_\perp},\ K_\parallel \in \mathcal{I}_{h_\parallel} \right\}.$$

In other words an element K of the partition $\mathcal{M}_h$ is a three-dimensional cube or prism and of course we have $\overline{\Omega} = \cup_{K \in \mathcal{M}_h} K$, $\overline{\Omega}_\perp = \cup_{K_\perp \in \mathcal{T}_{h_\perp}} K_\perp$ and $\overline{\Omega}_\parallel = \cup_{K_\parallel \in \mathcal{I}_{h_\parallel}} K_\parallel$. We define the mesh sizes $h_\perp > 0$, $h_\parallel > 0$ and $h > 0$, relative to the partitions, as usual:

$$h_\perp = \max_{K_\perp \in \mathcal{T}_{h_\perp}} \operatorname{diam}(K_\perp), \quad h_\parallel = \max_{K_\parallel \in \mathcal{I}_{h_\parallel}} \operatorname{diam}(K_\parallel) \quad \text{and} \quad h = \max(h_\perp, h_\parallel).$$

We define the set $\mathcal{E}_h$ as the set of edges of the partition $\mathcal{M}_h$. Since the partition $\mathcal{M}_h$ is built as the Cartesian product of perpendicular and parallel directions, we get $\mathcal{E}_h = \mathscr{E}_{h_\perp} \cup \mathcal{E}_{h_\parallel}$, where $\mathscr{E}_{h_\perp}$ and $\mathcal{E}_{h_\parallel}$ are defined by

$$\mathscr{E}_{h_\perp} = \bigcup_{K_\perp \in \mathcal{T}_{h_\perp}, K_\parallel \in \mathcal{I}_{h_\parallel}} (\partial K_\perp \times K_\parallel), \quad \mathcal{E}_{h_\parallel} = \bigcup_{K_\perp \in \mathcal{T}_{h_\perp}, K_\parallel \in \mathcal{I}_{h_\parallel}} (K_\perp \times \partial K_\parallel).$$

Finally, we have naturally the decomposition $\mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^\partial$, where $\mathcal{E}_h^0$ and $\mathcal{E}_h^\partial$ denote, respectively, the set of interior and boundary edges of the partition $\mathcal{M}_h$. Of course we have $\mathcal{E}_h^0 = \mathscr{E}_{h_\perp}^0 \cup \mathcal{E}_{h_\parallel}^0$ and $\mathcal{E}_h^\partial = \mathscr{E}_{h_\perp}^\partial \cup \mathcal{E}_{h_\parallel}^\partial$, with obvious definitions for $\mathscr{E}_{h_\perp}^0$, $\mathscr{E}_{h_\perp}^\partial$, $\mathcal{E}_{h_\parallel}^0$ and $\mathcal{E}_{h_\parallel}^\partial$.

Now, we need to define the trace operators 'average' $\{\!\{\cdot\}\!\}$ and 'jump' $[\![\cdot]\!]$. For this purpose, for every $K \in \mathcal{M}_h$ and for every function $\varphi \in H^s(K)$ $(s > 3/2)$, we define the traces of φ on the edges $\partial K_\perp \times K_\parallel$ and $K_\perp \times \partial K_\parallel$ as follows. Denoting by ν_K the outward normal vector to the element K, then for two adjacent cells K^ℓ and K^r (r denotes the right cell and ℓ the left one) of $\mathcal{M}_h$ and a point P of their common boundary, we set $\varphi^\alpha(P) = \lim_{\epsilon\to 0} \varphi^\alpha(P - \epsilon \nu_{K^\alpha})$, with $\alpha \in \{\ell, r\}$ and call these values the traces of φ from the interior of K^α. Let $K \in \mathcal{M}_h$ and $e \in \partial K \cap \mathcal{E}_h^0$, we then define the average trace operator $\{\!\{\cdot\}\!\} : H^s(K) \to H^{s-1/2}(e)$ $(s > 3/2)$, such that

$$\{\!\{\varphi\}\!\} = \frac{1}{2}\left(\varphi^\ell + \varphi^r\right) \quad \text{on } e \in \partial K \cap \mathcal{E}_h^0$$

and the jump trace operator $[\![\cdot]\!] : H^s(K) \to H^{s-1/2}(e)$ $(s > 3/2)$, such that

$$[\![\varphi]\!] = \varphi^\ell \nu_{K^\ell} + \varphi^r \nu_{K^r} \quad \text{on } e \in \partial K \cap \mathcal{E}_h^0.$$

The outward normal vector ν_K to a finite element $K = K_\perp \times K_\parallel$ can be decomposed along the perpendicular and parallel direction as $\nu_K = (\nu_K)_\perp + (\nu_K)_\parallel = (\nu_{K_\perp}^{\mathrm{T}}, 0)^{\mathrm{T}} + (0, 0, \nu_{K_\parallel})^{\mathrm{T}}$, where $\nu_{K_\perp}$ is the outward normal vector to the element $K_\perp \in \mathcal{T}_{h_\perp}$ and $\nu_{K_\parallel} = \pm 1$ when $\nu_{K_\perp} = 0$, and 0 when $\nu_{K_\perp} \neq 0$. Eventually, we also introduce the jump trace operators $[\![\cdot]\!]_\perp$ and $[\![\cdot]\!]_\parallel$, continuous maps from $H^s(K)$ to $H^{s-1/2}(e)$, defined as

$$[\![\varphi]\!]_\perp = \varphi^\ell \nu_{K_\perp^\ell} + \varphi^r \nu_{K_\perp^r} \quad \text{on } e \in \mathscr{E}_{h_\perp}^0, \qquad [\![\varphi]\!]_\parallel = \varphi^\ell \nu_{K_\parallel^\ell} + \varphi^r \nu_{K_\parallel^r} \quad \text{on } e \in \mathcal{E}_{h_\parallel}^0.$$

Finite element spaces. Let us first define the polynomial spaces $\mathbb{P}_k^d$, $\mathbb{Q}_k^d$ and $\mathbb{PR}_k$. Let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$. Using the multi-index notation $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d} \in \mathbb{R}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$, we then define $\mathbb{P}_k^d$ and $\mathbb{Q}_k^d$ such that

$$\mathbb{P}_k^d = \left\{ p(x) = \sum_{|\alpha| \le k} c_\alpha x^\alpha; \;\; c_\alpha \in \mathbb{R} \right\} \quad \text{and} \quad \mathbb{Q}_k^d = \left\{ p(x) = \sum_{\substack{0 \le \alpha_i \le k \\ 1 \le i \le d}} c_\alpha x^\alpha; \;\; c_\alpha \in \mathbb{R} \right\}.$$

We then define $\mathbb{PR}_k$ as

$$\mathbb{PR}_k = \left\{ p(x) = p_\perp(x_\perp) \otimes p(x_\parallel); \;\; p_\perp \in \mathbb{P}_k^2, \; p_\parallel \in \mathbb{P}_k^1 \right\}.$$

Let us notice that the set of polynomial $\mathbb{P}_k^1$ $(= \mathbb{Q}_k^1)$ can be orthogonalized (e.g., using the Gram–Schmidt algorithm) to obtain an orthogonal basis such that classical orthogonal polynomials (e.g., Legendre polynomials), and that $\mathbb{Q}_k^d = {\mathbb{Q}_k^1}^{\otimes^d}$.

We are now ready to introduce the finite element spaces V_h^k, Φ_h^k, $\widehat{\Phi}_h^k$, $\widehat{\Phi}_h^k$, $\bar{\Phi}_{h_\perp}^k$ and $X_{h_\parallel}^k$ defined as

$$\begin{aligned}
V_h^k &= \left\{ v \in L^2(\Omega) : \; v_{|K} \in \mathscr{P}_k(K), \; \forall K \in \mathcal{M}_h \right\}, \\
\widehat{\Phi}_h^k &= V_h^k \cap L^2\left(\Omega_\parallel; \mathscr{C}^0(\overline{\Omega}_\perp)\right) \subset L^2\left(\Omega_\parallel; \mathscr{C}^0(\overline{\Omega}_\perp) \cap H^1(\Omega_\perp)\right), \\
\widehat{\Phi}_h^k &= V_h^k \cap \mathscr{C}^0(\overline{\Omega}) \subset \mathscr{C}^0(\overline{\Omega}) \cap H^1(\Omega), \\
\bar{\Phi}_{h_\perp}^k &= \left\{ \varphi \in L^2(\Omega_\perp) : \; \varphi_{|K_\perp} \in \mathcal{P}_k(K_\perp), \; \forall K_\perp \in \mathcal{T}_{h_\perp} \right\} \cap \mathscr{C}^0(\overline{\Omega}_\perp) \subset \mathscr{C}^0(\overline{\Omega}_\perp) \cap H^1(\Omega_\perp), \\
X_{h_\parallel}^k &= \left\{ \varphi \in L^2(\Omega_\parallel) : \; v_{|K_\parallel} \in \mathbb{P}_k^1(K_\parallel), \; \forall K_\parallel \in \mathcal{I}_{h_\parallel} \right\},
\end{aligned}$$

where $\mathscr{P}_k$ (respectively $\mathcal{P}_k$) stands for either $\mathbb{Q}_k^3$ (respectively $\mathbb{Q}_k^2$) when the element K (respectively $K_\perp$) is a cube (respectively rectangle) or $\mathbb{PR}_k$ (respectively $\mathbb{P}_k^2$) when the element K (respectively $K_\perp$) is a prism (respectively triangle). Eventually, let the finite element space Φ_h^k be either $\widehat{\Phi}_h^k$ or $\widehat{\Phi}_h^k$.

3.1.3 *Projection and interpolation operators.* Here, we introduce some projection-interpolation operators associated to the finite element spaces defined in Section 3.1.2. We recall their stability and approximation properties, which are needed to perform the analysis of the scheme.

Notation. Let $\mathcal{D}$ be a bounded domain. We denote by $W^{m,p}(\Omega)$ the usual Sobolev spaces equipped with the norm $\|\cdot\|_{m,p,\Omega}$ and seminorm $|\cdot|_{m,p,\Omega}$, $m \in \mathbb{Z}$, $1 \le p \le \infty$, whose definitions and properties can be found, for example, in Adams (1975). When $m = 2$, we use the following standard short notation: $H^m(\Omega) = W^{m,p}(\Omega)$ for the Sobolev space, $\|\cdot\|_{m,\Omega} = \|\cdot\|_{m,2,\Omega}$ for the norm and $|\cdot|_{m,\Omega} = |\cdot|_{m,2,\Omega}$ for the seminorm. We also introduce the norm over the skeleton $\mathcal{E}_h$ of the partition $\mathcal{M}_h$: for $\varphi \in W^{m+1/p,p}(\Omega)$,

$$\|\varphi\|_{m,p,\mathcal{E}_h}^p = \sum_{e \in \mathcal{E}_h} \|\varphi\|_{m,p,e}^p, \quad 1 \le p < \infty; \qquad \|\varphi\|_{m,\infty,\mathcal{E}_h} = \sup_{e \in \mathcal{E}_h} \|\varphi\|_{m,\infty,e}.$$

Projection-interpolation operators. Let $k \ge 0$ and $\Pi_h : L^2(\Omega) \longrightarrow V_h^k$, be the standard L^2-projection operator onto V_h^k, which satisfies the following stability and approximation properties. The operator Π_h is stable in any L^p-norm—in L^2 by definition—(see Crouzeix & Thomée, 1987):

$$\|\Pi_h \varphi\|_{0,p,\Omega} \le C \|\varphi\|_{0,p,\Omega} \;\; \forall \varphi \in L^p(\Omega), \;\; 1 \le p \le \infty, \tag{3.1}$$

and is also locally $W^{1,p}$-stable (see Crouzeix & Thomée, 1987):

$$\|\Pi_h \varphi\|_{1,p,K} \le C \|\varphi\|_{1,p,K} \;\; \forall \varphi \in W^{1,p}(K), \;\; 1 \le p \le \infty, \;\; K \in \mathcal{M}_k. \tag{3.2}$$

The operator Π_h also satisfies a high-order approximation property in Sobolev spaces (see Ciarlet, 1991; Girault & Raviart, 1986):

$$\|\varphi - \Pi_h \varphi\|_{0,p,\Omega} + h^{1/p} \|\varphi - \Pi_h \varphi\|_{0,p,\mathcal{E}_h} \le C h^{k+1} |\varphi|_{k+1,p,\Omega} \;\; \forall \varphi \in W^{k+1,p}(\Omega), \;\; 1 \le p \le \infty, \tag{3.3}$$

where the constant C in (3.3) depends only on the polynomial degree k and geometric constants related to the shape regularity of the mesh $\mathcal{M}_h$. We have also the inverse inequality (see Ciarlet, 1991): for $0 \le p, q \le \infty$,

$$\|v\|_{1,p,\Omega} \le C h^{-1} \|v\|_{0,p,\Omega}, \quad \|v\|_{0,p,\Omega} \le C h^{-d \max(0, 1/q - 1/p)} \|v\|_{0,q,\Omega}, \;\; \forall v \in V_h^k, \tag{3.4}$$

where d denotes the dimension of the space ($d = 3$ for (V_h^k, Π_h), (V_h^0, π_h), $(\widehat{\Phi}_h^k, I_h)$, $(\widehat{\Phi}_h^k, \mathcal{P}_h)$; $d = 2$ for $(\bar{\Phi}_{h_\perp}^k, \mathbb{I}_{h_\perp})$; $d = 1$ for $(X_{h_\parallel}^k, \mathbb{P}_{h_\parallel})$; see below). We also need of the following trace estimate (see Ciarlet, 1991):

$$\|v\|_{0,p,\mathcal{E}_h} \leq C(p)h^{-1/p}\|v\|_{0,p,\Omega} \ \ \forall v \in V_h^k, \ \ 1 \leq p \leq \infty. \tag{3.5}$$

Of course, the global estimates (3.1)–(3.5) are also valid locally when one replaces Ω by any $K \in \mathcal{M}_h$ and $\mathcal{E}_h$ by any $e \in \partial K$, and one has in addition for all $K \in \mathcal{M}_h$,

$$|\varphi - \Pi_h\varphi|_{\ell,p,K} \leq Ch^{k+1-\ell}|\varphi|_{k+1,p,K} \ \ \forall \varphi \in W^{k+1,p}(K), \ \ 1 \leq p \leq \infty, \ \ 0 \leq \ell \leq k+1. \tag{3.6}$$

We set $\pi_h \equiv \Pi_h$ when $k = 0$, i.e., the L^2-projection on a constant function, and denote by $\mathbb{P}_{h_\parallel} : L^2(\Omega_\parallel) \longrightarrow X_{h_\parallel}^k$ the standard L^2-projection onto $X_{h_\parallel}^k$. We now introduce the interpolation operator $I_h : W^{m,p}(\Omega) \cap \mathscr{C}^0(\overline{\Omega}) \longrightarrow \widehat{\Phi}_h^k$, with $m > 3/p$ (respectively $\mathbb{I}_{h_\perp} : W^{m,p}(\Omega_\perp) \cap \mathscr{C}^0(\overline{\Omega}_\perp) \longrightarrow \bar{\Phi}_{h_\perp}^k$, with $m > 2/p$) defined on each element $K \in \mathcal{M}_h$ (respectively $K_\perp \in \mathcal{T}_{h_\perp}$) by

$$I_h\varphi_{|K} \in \mathscr{P}_k, \ \ I_h\varphi(x) = \varphi(x) \ \ \forall x \in \Sigma_k \quad \left(\text{respectively } \ \mathbb{I}_{h_\perp}\varphi_{|K_\perp} \in \mathcal{P}_k, \ \ \mathbb{I}_{h_\perp}\varphi(x) = \varphi(x) \ \ \forall x \in \Sigma_k^\perp\right).$$

Here, Σ_k (respectively $\Sigma_k^\perp$) denotes the regular principal lattice of order k associated to a finite element K, which is either a cube (respectively rectangle) or a prism (respectively a triangle) (see Ciarlet, 1991 for more details). Finally, we introduce the projection operator $\mathcal{P}_h : L^2\left(\Omega_\parallel; \mathscr{C}^0(\overline{\Omega}_\perp) \cap H^1(\Omega_\perp)\right) \cap W^{m,p}(\Omega) \longrightarrow \widehat{\Phi}_h^k$, defined as

$$\mathcal{P}_h = \mathbb{I}_{h_\perp} \otimes \mathbb{P}_{h_\parallel}.$$

The pairs formed by a finite element space and a reconstruction operator such as (V_h^0, π_h), $(\widehat{\Phi}_h^k, I_h)$, $(\widehat{\Phi}_h^k, \mathcal{P}_h)$, $(\bar{\Phi}_{h_\perp}^k, \mathbb{I}_{h_\perp})$ and $(X_{h_\parallel}^k, \mathbb{P}_{h_\parallel})$ satisfy stability, approximation, inverse and trace estimates similar to (3.1)–(3.6) (see Ciarlet, 1991). Moreover, the interpolation operator I_h (respectively $\mathbb{I}_{h_\perp}$) satisfies additional stability properties (see Ciarlet, 1991): for $1 \leq p \leq \infty$ and $0 \leq \ell \leq 1$,

$$\begin{aligned} &|I_h\varphi|_{\ell,p,\Omega} \leq C|\varphi|_{\ell,p,\Omega} \ \ \forall \varphi \in W^{m,p}(\Omega) \cap \mathscr{C}^0(\overline{\Omega}), \ \ m > 3/p \\ &\left(\text{respectively } \ |\mathbb{I}_{h_\perp}\varphi|_{\ell,p,\Omega_\perp} \leq C|\varphi|_{\ell,p,\Omega_\perp} \ \ \forall \varphi \in W^{m,p}(\Omega_\perp) \cap \mathscr{C}^0(\overline{\Omega}_\perp), \ \ m > 2/p\right), \end{aligned} \tag{3.7}$$

and approximation properties (see Ciarlet, 1991): for $1 \leq p \leq \infty$ and $0 \leq \ell \leq 1$,

$$\begin{aligned} &|\varphi - I_h\varphi|_{\ell,p,\Omega} \leq Ch^{k+1-\ell}|\varphi|_{k+1,p,\Omega} \ \ \forall \varphi \in W^{k+1,p}(\Omega) \cap \mathscr{C}^0(\overline{\Omega}) \\ &\left(\text{respectively } \ |\varphi - \mathbb{I}_{h_\perp}\varphi|_{\ell,p,\Omega_\perp} \leq Ch_\perp^{k+1-\ell}|\varphi|_{k+1,p,\Omega_\perp} \ \ \forall \varphi \in W^{k+1,p}(\Omega_\perp) \cap \mathscr{C}^0(\overline{\Omega}_\perp)\right). \end{aligned} \tag{3.8}$$

By (3.7) and (3.8), and stability-approximation properties of $\mathbb{P}_{h_\parallel}$, we deduce that the operator $\mathcal{P}_h$ is a continuous map from $L^2\left(\Omega_\parallel; \mathscr{C}^0(\overline{\Omega}_\perp) \cap H^1(\Omega_\perp)\right) \cap W^{m,p}(\Omega)$ to $L^2\left(\Omega_\parallel; \mathscr{C}^0(\overline{\Omega}_\perp) \cap H^1(\Omega_\perp)\right)$ and satisfies the following stability approximation properties: for $1 \leq p \leq \infty$ and $0 \leq \ell \leq 1$,

$$\|\mathcal{P}_h\varphi\|_{L^2(\Omega_\|;\,W^{\ell,p}(\Omega_\perp))} \leq C\|\varphi\|_{L^2(\Omega_\|;\,W^{\ell,p}(\Omega_\perp))} \quad \forall\varphi \in L^2\left(\Omega_\|;\, \mathscr{C}^0(\overline{\Omega}_\perp) \cap H^1(\Omega_\perp)\right) \cap W^{m,p}(\Omega),$$
$$\|\varphi - \mathcal{P}_h\varphi\|_{L^2(\Omega_\|;\,W^{\ell,p}(\Omega_\perp))} \leq Ch^{k+1-\ell}|\varphi|_{k+1,\Omega} \quad \forall\varphi \in L^2\left(\Omega_\|;\, \mathscr{C}^0(\overline{\Omega}_\perp) \cap H^1(\Omega_\perp)\right) \cap W^{k+1,p}(\Omega).$$

Eventually, let the reconstruction operator $\mathcal{R}_h$ be either $\mathcal{P}_h$ or I_h. We then have for $1 \leq p \leq \infty$, and $0 \leq \ell \leq k+1$,

$$\|\varphi - \mathcal{R}_h\varphi\|_{\ell,p,K} \leq Ch^{k+1-\ell}|\varphi|_{\ell,k+1,K} \quad \forall\varphi \in W^{k+1,p}(K), \quad K \in \mathcal{M}_h.$$

3.1.4 *Gyroaverage operator.* Let us now state some obvious stability properties of the gyroaverage operator $\mathcal{J}_\perp$, which will be useful for the analysis of the numerical scheme. From (2.3), it is clear that the gyroaverage operator $\mathcal{J}_\perp$ is a continuous map from $W^{m,\infty}(\Omega_\perp)$ into itself, with $m \geq 0$, and one has

$$|\mathcal{J}_\perp\varphi|_{m,\infty,\Omega_\perp} \leq |\varphi|_{m,\infty,\Omega_\perp} \quad \forall\varphi \in W^{m,\infty}(\Omega_\perp).$$

By (2.4), and since $J_0(|\eta|) \sim |\eta|^{-1/2}$ when $|\eta|$ goes to infinity, it is clear that the gyroaverage operator $\mathcal{J}_\perp$ is a continuous map from $H^m(\mathbb{R}^2)$ to $H^{m+1/2}(\mathbb{R}^2)$, with $m \geq 0$, and one has

$$|\mathcal{J}_\perp\varphi|_{m+1/2,\mathbb{R}^2} \leq |\varphi|_{m,\mathbb{R}^2} \quad \forall\varphi \in H^m(\mathbb{R}^2).$$

3.2 *Numerical methods*

Here, we expose the numerical scheme designed from discontinuous Galerkin methods. For this purpose, we use the Dirichlet boundary condition described in Section 3.1.1 with $\phi_b = 0$. First, we perform the semidiscretization in space of the contour equations (2.8) in Section 3.2.1, and second we discretize the quasi-neutrality equation (2.9) in Section 3.2.2.

3.2.1 *Discontinuous Galerkin approximation of contour equations.* In this section, we present a discontinuous Galerkin method to discretize the contour equations (2.8), which gives the approximation of contours from knowledge of the electrical potential approximation (see Section 3.2.2).

Let $k' \geq 0$ and $\phi_h \in \Phi_h^{k'}$ be the finite element approximation of the electrical potential, which is supposed given and whose computation will be specified later on (see Section 3.2.2). Let us note that when we use homogeneous Dirichlet conditions or periodic conditions in the transverse direction, the gyroaverage operator $\mathcal{J}_\perp$ is well defined by (2.3) since we can extend naturally (by periodicity, zero or uniform constant) the electrical potential approximation ϕ_h and the approximation of contours $v_{\sigma h}^\pm$. Let us remark that the integral (2.3) can be computed exactly since it involves only polynomials of cosines and sines. We observe that $\mathcal{J}_\perp\phi_h \notin \Phi_h^{k'}$ but of course $\mathcal{R}_h\mathcal{J}_\perp\phi_h$ is. Since $\mathcal{R}_h\mathcal{J}_\perp\phi_h \in \Phi_h^{k'}$ and $v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h) \cdot \nu_K = v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h) \cdot \nu_{K_\perp} = \nabla_\perp\mathcal{R}_h\mathcal{J}_\perp\phi_h \cdot \tau_{K_\perp}$ (with $\tau_{K_\perp} = \nu_{K_\perp}^\perp$), then the normal trace of the drift velocity $v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h) \cdot \nu_K$ is continuous across the boundary ∂K of any element $K \in \mathcal{M}_h$.

With $k \geq 0$, the discontinuous Galerkin approximation of (2.8) then reads, find $v^{\pm}_{\sigma h} \in V^h_k$ with $\sigma \in \mathfrak{L}^{\mathrm{NM}}$, such that for any $K \in \mathcal{M}_h$, and $\varphi_h \in V^k_h$, we have

$$\begin{aligned}\int_K \partial_t v^{\pm}_{\sigma h}\varphi_h \,\mathrm{d}x - \int_K v^{\pm}_{\sigma h} v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nabla_\perp\varphi_h\,\mathrm{d}x \\ + \int_{\partial K}\left\langle\!\left\langle v^{\pm}_{\sigma h} v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nu_{K_\perp}\right\rangle\!\right\rangle \varphi^{\ell}_h\,\mathrm{d}\gamma - \int_K\left(\mathcal{R}_h\mathcal{J}_\perp\phi_h + {v^{\pm}_{\sigma h}}^2/2\right)\partial_\parallel\varphi_h\,\mathrm{d}x \\ + \int_{\partial K}\left\langle\!\left\langle \mathcal{R}_h\mathcal{J}_\perp\phi_h\nu_{K_\parallel}\right\rangle\!\right\rangle\varphi^{\ell}_h\,\mathrm{d}\gamma + \int_{\partial K}\left\langle\!\left\langle\left({v^{\pm}_{\sigma h}}^2/2\right)\nu_{K_\parallel}\right\rangle\!\right\rangle\varphi^{\ell}_h\,\mathrm{d}\gamma = 0. \end{aligned} \tag{3.9}$$

In (3.9) it remains to define the numerical fluxes $\langle\!\langle\cdot\rangle\!\rangle$ in terms of average $\{\!\{\cdot\}\!\}$ and jump $[\![\cdot]\!]$ trace operators.

Numerical flux definitions. For the transverse flux term we choose a Lax–Friedrichs flux, with different diffusion parameters $\mathcal{D}_\perp$, allowing us to recover the upwind flux. Let $K \in \mathcal{M}_h$ and $e \in \mathcal{E}^0_h \cap \partial K$; we then define the numerical flux $\left\langle\!\left\langle v^{\pm}_{\sigma h} v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nu_{K_\perp}\right\rangle\!\right\rangle$ on the edge $e \in \mathscr{E}^0_{h_\perp}$ as

$$\begin{aligned}\left\langle\!\left\langle v^{\pm}_{\sigma h} v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nu_{K_\perp}\right\rangle\!\right\rangle_{|e} &= \{\!\{v^{\pm}_{\sigma h}\}\!\} v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nu_{K_\perp} + \frac{\mathcal{D}_{\perp,e}}{2}[\![v^{\pm}_{\sigma h}]\!]_\perp\cdot\nu_{K_\perp} \\ &= \left\langle\!\left\langle v^{\pm}_{\sigma h} v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\right\rangle\!\right\rangle_{|e}\cdot\nu_{K_\perp},\end{aligned} \tag{3.10}$$

where, with $S = e,\ \partial K,\ \partial K \cup \partial K'\ (K \cap K' \neq \emptyset),\ \ldots$, we set

$$\mathcal{D}_{\perp,e} = \begin{cases} \text{local} & : \ \max\limits_{x\in S} |v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nu_{K_\perp}|, \\ \text{global} & : \ \max\limits_{x\in e,\ e\in\mathcal{E}^0_h} |v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nu_{K_\perp}|, \\ \text{upwind} & : \ |v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nu_{K_\perp}|. \end{cases}$$

For the parallel flux term we choose a Lax-Friedrichs flux, with different diffusion parameters $\mathcal{D}^{\pm}_{\parallel\sigma}$ allowing us to recover equivalently Godunov (G), Engquist–Osher (EO), Roe (R) and upwind (U) fluxes. Let $K \in \mathcal{M}_h$ and $e \in \mathcal{E}^0_h \cap \partial K$; we then define the numerical flux $\left\langle\!\left\langle\left({v^{\pm}_{\sigma h}}^2/2\right)\nu_{K_\parallel}\right\rangle\!\right\rangle$ on the edge $e \in \mathcal{E}^0_{h_\parallel}$ as

$$\left\langle\!\left\langle\left({v^{\pm}_{\sigma h}}^2/2\right)\nu_{K_\parallel}\right\rangle\!\right\rangle_{|e} = \left\{\!\!\left\{\frac{{v^{\pm}_{\sigma h}}^2}{2}\right\}\!\!\right\}\cdot\nu_{K_\parallel} + \frac{\mathcal{D}^{\pm}_{\parallel\sigma,e}}{2}[\![v^{\pm}_{\sigma h}]\!]_\parallel\cdot\nu_{K_\parallel} = \left\langle\!\left\langle{v^{\pm}_{\sigma h}}^2/2\right\rangle\!\right\rangle_{|e}\cdot\nu_{K_\parallel}, \tag{3.11}$$

where, with $S = \{x\},\ e,\ \partial K,\ \partial K \cup \partial K'\ (K \cap K' \neq \emptyset),\ \ldots$, we set

$$\mathcal{D}^{\pm}_{\parallel\sigma,e} = \begin{cases} \text{local} & : \ \max\left\{|v|;\ \min\limits_{x\in S}\left(v^{\pm,\ell}_{\sigma h}, v^{\pm,r}_{\sigma h}\right) \leq v \leq \max\limits_{x\in S}\left(v^{\pm,\ell}_{\sigma h}, v^{\pm,r}_{\sigma h}\right)\right\}, \\ \text{global} & : \ \max\left\{|v|;\ \inf v^{\pm}_{\sigma h} \leq v \leq \sup v^{\pm}_{\sigma h}\right\}, \\ \text{(G), (EO), (R), (U)} & : \ \pm\left(v^{\pm,\ell}_{\sigma h} + v^{\pm,r}_{\sigma h}\right)/2. \end{cases}$$

Eventually, for the electrical potential flux term we choose an α-average flux term defined on the edge $e \in \mathcal{E}^0_{h_\parallel}$ by

$$\left\langle\!\!\left\langle \mathcal{R}_h \mathcal{J}_\perp \phi_h \nu_{K_\parallel} \right\rangle\!\!\right\rangle_{|e} = \left\{\!\!\left\{ \mathcal{R}_h \mathcal{J}_\perp \phi_h \right\}\!\!\right\}_\alpha \cdot \nu_{K_\parallel} = \left(\alpha(\mathcal{R}_h \mathcal{J}_\perp \phi_h)^\ell + (1-\alpha)(\mathcal{R}_h \mathcal{J}_\perp \phi_h)^r\right) \cdot \nu_{K_\parallel}$$
$$= \langle\!\langle \mathcal{R}_h \mathcal{J}_\perp \phi_h \rangle\!\rangle_{|e} \cdot \nu_{K_\parallel} \qquad (3.12)$$

with $\alpha \in [0, 1]$.

3.2.2 *Finite element approximations of the quasi-neutrality equation.* Here, we present two finite element methods to discretize the quasi-neutrality equation (2.9), which give the approximation of the electrical potential from the knowledge of the source term involving the approximation of all the contours (see Section 3.2.1).

Let $k \geq 0$, and $v^{\pm}_{\sigma h} \in V^h_k$ with $\sigma \in \mathfrak{L}^{\mathrm{NM}}$, given by the Discontinuous Galerkin Finite Element Method (DGFEM) scheme (3.9). We note that $\mathcal{J}_\perp v^{\pm}_{\sigma h} \notin V^h_k$ but of course $\Pi_h \mathcal{J}_\perp v^{\pm}_{\sigma h}$ is. We then define the charge density $\rho_h \in V^h_k$ and $\bar{\rho}_h \in L^2(\Omega_\perp)$ by

$$\rho_h = \sum_{\sigma \in \mathfrak{L}^{\mathrm{NM}}} \mathcal{A}_\sigma (v^+_{\sigma h} - v^-_{\sigma h})$$

and

$$\bar{\rho}_h = \langle \Pi_h \mathcal{J}_\perp \rho_h \rangle_\parallel - n_{i0} = \frac{1}{|\Omega_\parallel|} \int_{\Omega_\parallel} \Pi_h \mathcal{J}_\perp \rho_h \,\mathrm{d}x_\parallel - n_{i0}.$$

We also introduce the density $\tilde{\rho}_h$ defined by

$$\tilde{\rho}_h = \Pi_h \mathcal{J}_\perp \rho_h + b_0 \bar{\phi}_h - n_{i0}.$$

Here $b_0 = b_0(x_\perp) = e\tau n_{i0}/(k_B T_{i0}) \geq b^0_0 > 0$ (b^0_0 constant) and $\bar{\phi}_h \in \bar{\Phi}^{k'}_{h_\perp}$ satisfies the following variational discrete problem: find $\bar{\phi}_h \in \bar{\Phi}^{k'}_{h_\perp} \cap H^1_0(\Omega_\perp)$ such that

$$a_\perp(\bar{\phi}_h, \bar{\psi}_h) = L_{\perp h}(\bar{\psi}_h) \quad \forall \bar{\psi}_h \in \bar{\Phi}^{k'}_{h_\perp} \cap H^1_0(\Omega_\perp), \qquad (3.13)$$

with $a_0 = a_0(x_\perp) = n_{i0}/(B_0 \Omega_0) \geq a^0_0 > 0$ (a^0_0 constant). The bilinear form $a_\perp(\cdot, \cdot)$ and the linear form $L_{\perp h}(\cdot)$ are defined by

$$a_\perp(\bar{\phi}, \bar{\psi}) = \int_{\Omega_\perp} a_0 \nabla_\perp \bar{\phi} \cdot \nabla_\perp \bar{\psi} \,\mathrm{d}x \quad \text{and} \quad L_{\perp h}(\bar{\psi}) = \int_{\Omega_\perp} \bar{\rho}_h \bar{\psi} \,\mathrm{d}x, \qquad \bar{\phi},\, \bar{\psi} \in H^1_0(\Omega_\perp).$$

Eventually, the approximation ϕ_h, of the electrical potential ϕ satisfying (2.9), is obtained by solving the following variational problem: find $\phi_h \in \Phi^{k'}_h \cap L^2\left(\Omega_\parallel; H^1_0(\Omega_\perp)\right)$ such that

$$a(\phi_h, \psi_h) = L_h(\psi_h) \quad \forall \psi_h \in \Phi^{k'}_h \cap L^2\left(\Omega_\parallel; H^1_0(\Omega_\perp)\right), \qquad (3.14)$$

where the bilinear form $a(\cdot,\cdot)$ and the linear form $L_h(\cdot)$ are defined by

$$a(\phi,\psi)=\int_{\Omega} a_0\nabla_\perp\phi\cdot\nabla_\perp\psi\,\mathrm{d}x+\int_{\Omega} b_0\phi\psi\,\mathrm{d}x \quad \text{and} \quad L_h(\psi)=\int_{\Omega}\tilde{\rho}_h\psi\,\mathrm{d}x, \quad \phi,\,\psi\in L^2\left(\Omega_\parallel;\,H^1_0(\Omega_\perp)\right).$$

Let us note that on the one hand the bilinear forms $a_\perp(\cdot,\cdot)$ and $a(\cdot,\cdot)$ are, respectively, coercive and continuous in $\bar{\Phi}^{k'}_{h_\perp}\cap H^1_0(\Omega_\perp)$ and $\Phi^{k'}_h\cap L^2\left(\Omega_\parallel;\,H^1_0(\Omega_\perp)\right)$, and on the other hand, the linear forms $L_{\perp h}(\cdot)$ and $L_h(\cdot)$ are, respectively, continuous in $\bar{\Phi}^{k'}_{h_\perp}\cap H^1_0(\Omega_\perp)$ and $\Phi^{k'}_h\cap L^2\left(\Omega_\parallel;\,H^1_0(\Omega_\perp)\right)$. Therefore, using the Lax–Milgram theorem, the discrete problem (3.13) (respectively (3.14)) has a unique solution $\bar{\phi}_h\in\bar{\Phi}^{k'}_{h_\perp}\cap H^1_0(\Omega_\perp)$ (respectively $\phi_h\in\Phi^{k'}_h\cap L^2\left(\Omega_\parallel;\,H^1_0(\Omega_\perp)\right)$).

4. Convergence and error analysis

Here, we prove the convergence of the DGFEM scheme, which is constituted by (3.9) and (3.13) and (3.14), and described in Section 3.2. We also obtain high-order error estimates for the contour and electrical potential approximations. First, we start by presenting the main theorem in Section 4.1. Second, in Section 4.2 we formulate two lemmas, which give L^2- and L^∞-error estimates for the electrical potential approximation in terms of the small parameter h and L^2-error estimates for the approximation of contours. Using these estimates, for the contour approximations, we obtain a final error estimate, which is obtained through several lemmas in Section 4.3. In Section 4.4, we give some properties related to the L^2- and L^∞-stability of the approximate solution.

In the sequel, in order to lighten the notation without introducing any confusion, we sometimes omit to write the set over which contour indices $(\beta,\sigma)\in\{-,+\}\times\mathfrak{L}^{\mathrm{NM}}$ run.

4.1 *Main theorem*

The main result of this article is the theorem that gives the convergence and high-order estimates for the DGFEM scheme (3.9) and (3.13)–(3.14), described in Section 3.2.

THEOREM 4.1 Assume $k\geq 5/2$ and $k'\geq 5/2$. Let $(v^\pm_\sigma,\phi)\in\mathrm{Lip}\left(0,T;\,H^{k+1}(\Omega)\right)\cap L^\infty\left(0,T;\,H^{k+1}(\Omega)\right.$ $\left.\cap W^{1,\infty}(\Omega)\right)\times L^\infty\left(0,T;\,H^{k'+1}(\Omega)\cap W^{2,\infty}(\Omega)\cap L^2\left(\Omega_\parallel;\,H^1_0(\Omega_\perp)\right)\right)$ be the compactly supported solution at time $t\in[0,T)$ of the gyrokinetic-waterbag system (2.8)–(2.9). Let $(v^\pm_{\sigma h},\phi_h)\in V^k_h\times\Phi^{k'}_h\cap L^2\left(\Omega_\parallel;\,H^1_0(\Omega_\perp)\right)$ be the approximate solution of (2.8) and (2.9) given by the DGFEM scheme (3.9) and (3.13) and (3.14) described in Section 3.2. Then following estimates hold:

$$\left\|v^\pm_\sigma-v^\pm_{\sigma h}\right\|_{L^\infty\left(0,T;L^2(\Omega)\right)}\leq C_\star h^{\min(k,k')}\quad\forall\sigma\in\mathfrak{L}^{\mathrm{NM}},$$

$$\|\phi-\phi_h\|_{L^\infty\left(0,T;L^2(\Omega)\right)}\leq C_\star h^{\min(k,k')+1}.$$

Here, the constant $C_\star$ depends on the final time T, the polynomial degrees k and k', some geometric constants related to the shape regularity of the mesh $\mathcal{M}_h$, a_0, b_0, Ω, $\sum_\sigma\mathcal{A}_\sigma$ and the solution $(v^\pm_\sigma,\phi)$ through the norms $\|\phi\|_{L^\infty\left(0,T;H^{k'+1}(\Omega)\right)}$, $\|\phi\|_{L^\infty\left(0,T;W^{2,\infty}(\Omega)\right)}$ and $\sum_{\beta,\sigma}\mathcal{A}_\sigma C\big(\|\partial_t v^\beta_\sigma\|_{L^\infty\left(0,T;H^{k+1}(\Omega)\right)}$, $\|v^\beta_\sigma\|_{L^\infty\left(0,T;H^{k+1}(\Omega)\right)}$, $\|v^\beta_\sigma\|_{L^\infty\left(0,T;W^{1,\infty}(\Omega)\right)}\big)$.

REMARK 4.2 The order of convergence $\min(k,k')$ is suboptimal with respect to the order $\min(k,k')+1$ expected when we use polynomial reconstructions of degree k (for contours) and k' (for the electrical potential). This restriction of the convergence order comes from the source term $\partial_{\parallel}\phi$ of (2.8), which is independent of the contours $v_{\sigma}^{\pm}$ unlike the transport terms, i.e., the perpendicular incompressible Euler term and the parallel Burgers term. The gyrokinetic-waterbag equations (2.8) and (2.9) and the numerical DGFEM scheme that follows have here the form of a quasilinear weakly coupled system, whereas, due to the absence of a parallel derivative in the quasi-neutrality equation (2.9), the system (2.8)–(2.9) is a strongly (algebraically) coupled system. Therefore, such a system should be rewritten as a true nonlinear system of conservation laws with nonlocal fluxes, i.e., by replacing the electrical potential by a nonlocal (respectively local) operator in the perpendicular (respectively parallel) direction acting over all the contours. A numerical method designed from a true system of conservation laws may recover the standard order of convergence with the drawbacks of a loss of a natural decoupling between contours and thus computational efficiency. Some perspectives are evoked in Remarks 4.7, 4.8, 4.10, and 4.12 to recover the optimal order of convergence $\min(k, k') + 1$.

4.2 *Error estimates for the electrical potential*

Here, we state the error estimates in L^2- and L^∞-norms for the approximation of the electrical potential whose proofs are postponed to Appendix A. We start with L^2-error estimates.

PROPOSITION 4.3 Assume $k' \geq 1, k \geq 0$ and $\phi \in L^\infty\big(0, T;\, H^{k'+1}(\Omega)\big)$ is the unique solution of the quasi-neutrality equation (2.9). Let $\phi_h \in L^\infty\big(0, T;\, \Phi_h^{k'} \cap L^2\big(\Omega_\parallel;\, H_0^1(\Omega_\perp)\big)\big)$ be the approximation of ϕ given by the unique solution of the discrete problem (3.13) and (3.14) with the given functions $v_{\sigma h}^{\pm} \in L^\infty\big(0, T;\, V_h^k\big)$, $\sigma \in \mathfrak{L}^{\mathrm{NM}}$. Then there exists a constant C_2 depending on a_0, b_0, Ω, $\sum_\sigma \mathcal{A}_\sigma$, $\|\phi\|_{L^\infty\left(0,T;\,H^{k'+1}(\Omega)\right)}$ and $\sum_{\beta,\sigma} \mathcal{A}_\sigma C\big(\|v_\sigma^\beta\|_{L^\infty\left(0,T;\,H^{k+1}(\Omega)\right)}\big)$ such that for all $t \in [0, T)$ and $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq 1$,

$$\|\partial_\perp^\alpha(\phi(t) - \phi_h(t))\|_{0,\Omega} \leq C_2 h^{\min(k+|\alpha|,k')+1-|\alpha|} + C_2 h^{1-|\alpha|}\left(\sum_{\beta,\sigma} \mathcal{A}_\sigma \left\|v_\sigma^\beta(t) - v_{\sigma h}^\beta(t)\right\|_{0,\Omega}^2\right)^{1/2}. \tag{4.1}$$

The forthcoming error analysis also needs L^∞-error estimates for the approximation of the electrical potential ϕ_h.

PROPOSITION 4.4 Assume $k' \geq 1$, $k \geq 1$ and $\phi \in L^\infty\big(0, T;\, W^{k'+1,\infty}(\Omega)\big)$ is the unique solution of the quasi-neutrality equation (2.9). Let $\phi_h \in L^\infty\big(0, T;\, \Phi_h^{k'} \cap L^2\big(\Omega_\parallel;\, H_0^1(\Omega_\perp)\big)\big)$ be the approximation of ϕ given by the unique solution of the discrete problem (3.13)–(3.14) with the given functions $v_{\sigma h}^{\pm} \in L^\infty\big(0, T;\, V_h^k\big)$, $\sigma \in \mathfrak{L}^{\mathrm{NM}}$. Then there exists a constant C_∞ depending on a_0, b_0, Ω, $\sum_\sigma \mathcal{A}_\sigma$, $\|\phi\|_{L^\infty\left(0,T;\,H^{k'+1}(\Omega)\right)}$, $\|\phi\|_{L^\infty\left(0,T;\,W^{k'+1,\infty}(\Omega)\right)}$ and $\sum_{\beta,\sigma} \mathcal{A}_\sigma C\left(\|v_\sigma^\beta\|_{L^\infty\left(0,T;\,H^{k+1}(\Omega)\right)}\right)$ such that for all $t \in [0, T)$ and $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq 1$,

$$\begin{aligned}\|\partial_\perp^\alpha(\phi(t) - \phi_h(t))\|_{0,\infty,\Omega} \leq{}& C_\infty h^{\min(k,k')+1-|\alpha|}|\ln h|^{(1-|\alpha|)\varkappa}\\ &+ C_\infty h^{1/2-|\alpha|}|\ln h|\left(\sum_{\beta,\sigma} \mathcal{A}_\sigma \left\|v_\sigma^\beta(t) - v_{\sigma h}^\beta(t)\right\|_{0,\Omega}^2\right)^{1/2},\end{aligned} \tag{4.2}$$

where $\varkappa = 1$ if $k' = 1$ and $\varkappa = 0$ if $k' > 1$.

4.3 *Proof of the main theorem*

The proof of Theorem 4.1 given here is decomposed into four parts. The first part, in Section 4.3.1, concerns the perpendicular error term estimate. The second part, in Section 4.3.2, is devoted to the parallel error term coming from the electrical potential, while the third part, in Section 4.3.2, concerns the parallel error term coming from contours. Finally, Section 4.3.4 completes the proof. We first start by giving the error equation and by estimating the 'time-derivative' error.

Multiplying the contour equations (2.8) by a test function $\varphi \in L^2(\Omega)$ (such that for all $K \in \mathcal{M}_h$, $\varphi_{|K} \in H^s(K)$ with $s > 3/2$), integrating the result over any cell $K \in \mathcal{M}_h$ and performing some integration by parts, we then obtain

$$\begin{aligned}\int_K \partial_t v_\sigma^\pm \varphi \, \mathrm{d}x - \int_K v_\sigma^\pm v_E(\mathcal{J}_\perp \phi) \cdot \nabla_\perp \varphi \, \mathrm{d}x \\ + \int_{\partial K} v_\sigma^\pm v_E(\mathcal{J}_\perp \phi) \cdot \nu_{K_\perp} \varphi^\ell \, \mathrm{d}\gamma - \int_K \left(\mathcal{J}_\perp \phi + {v_\sigma^\pm}^2/2\right) \partial_\parallel \varphi \, \mathrm{d}x \\ + \int_{\partial K} \mathcal{J}_\perp \phi \nu_{K_\parallel} \varphi^\ell \, \mathrm{d}\gamma + \int_{\partial K} \left({v_\sigma^\pm}^2/2\right) \nu_{K_\parallel} \varphi^\ell \, \mathrm{d}\gamma = 0.\end{aligned} \tag{4.3}$$

Let us introduce the notation

$$e_{v_\sigma^\pm} = v_\sigma^\pm - v_{\sigma h}^\pm = (\Pi_h v_\sigma^\pm - v_{\sigma h}^\pm) - (\Pi_h v_\sigma^\pm - v_\sigma^\pm) = \xi_{v_\sigma^\pm} - \eta_{v_\sigma^\pm}.$$

Taking $\varphi = \varphi_h \in V_h^{k'}$ in the variational formulation (4.3), subtracting the result from the numerical scheme (3.9) and finally taking $\varphi_h = \xi_{v_\sigma^\pm}$, we then obtain the following error equation:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_K |\xi_{v_\sigma^\pm}|^2 \mathrm{d}x = T_{tK}^{\pm,\sigma} + T_{\perp K}^{\pm,\sigma} + T_{\parallel K}^{\pm,\sigma}. \tag{4.4}$$

Here, we have the definitions

$$T_{tK}^{\pm,\sigma} = \int_K \xi_{v_\sigma^\pm} \partial_t \eta_{v_\sigma^\pm} \, \mathrm{d}x, \tag{4.5}$$

$$\begin{aligned}T_{\perp K}^{\pm,\sigma} &= \int_K \left(v_\sigma^\pm v_E(\mathcal{J}_\perp \phi) - v_{\sigma h}^\pm v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h)\right) \cdot \nabla_\perp \xi_{v_\sigma^\pm} \, \mathrm{d}x \\ &\quad - \int_{\partial K} \left(v_\sigma^\pm v_E(\mathcal{J}_\perp \phi) - \left\langle\!\left\langle v_{\sigma h}^\pm v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h)\right\rangle\!\right\rangle\right) \cdot \nu_{K_\perp} \xi_{v_\sigma^\pm}^\ell \, \mathrm{d}\gamma\end{aligned} \tag{4.6}$$

and

$$T_{\parallel K}^{\pm,\sigma} = T_{\parallel K, v}^{\pm,\sigma} + T_{\parallel K, \phi}^{\pm,\sigma}, \tag{4.7}$$

where

$$T_{\parallel K, v}^{\pm,\sigma} = \int_K \left({v_\sigma^\pm}^2/2 - {v_{\sigma h}^\pm}^2/2\right) \partial_\parallel \xi_{v_\sigma^\pm} \, \mathrm{d}x - \int_{\partial K} \left(\left({v_\sigma^\pm}^2/2\right) - \left\langle\!\left\langle {v_{\sigma h}^\pm}^2/2 \right\rangle\!\right\rangle\right) \nu_{K_\parallel} \xi_{v_\sigma^\pm}^\ell \, \mathrm{d}\gamma \tag{4.8}$$

and

$$T^{\pm,\sigma}_{\parallel K,\phi} = \int_K (\mathcal{J}_\perp \phi - \mathcal{R}_h \mathcal{J}_\perp \phi_h)\, \partial_\parallel \xi_{v^\pm_\sigma}\, \mathrm{d}x - \int_{\partial K} (\mathcal{J}_\perp \phi - \langle\!\langle \mathcal{R}_h \mathcal{J}_\perp \phi_h \rangle\!\rangle)\, \nu_{K_\parallel} \xi^\ell_{v^\pm_\sigma}\, \mathrm{d}\gamma. \tag{4.9}$$

The control of the 'time-derivative' error term (4.5) is given by the following lemma.

LEMMA 4.5 There exists a constant C such that the 'time-derivative' error term is bounded as

$$\sum_{\beta,\sigma} \sum_{K\in\mathcal{M}_h} \mathcal{A}_\sigma T^{\beta,\sigma}_{tK} \le C\Big(h^{2k+2} + \sum_{\beta,\sigma} \mathcal{A}_\sigma \|\xi_{v^\beta_\sigma}\|^2_{0,\Omega}\Big),$$

where the constant C depends on $\sum_{\beta,\sigma} \mathcal{A}_\sigma \|\partial_t \eta_{v^\beta_\sigma}\|^2_{L^\infty(0,T;H^{k+1}(\Omega))}$.

Proof. Using the Cauchy–Schwarz estimate, the Young inequality and approximation properties of the L^2-projection operator Π_h (Section 3.1.3), we obtain

$$\begin{aligned}\sum_{K\in\mathcal{M}_h} T^{\pm,\sigma}_{tK} &= \sum_{K\in\mathcal{M}_h} \int_K \xi_{v^\pm_\sigma} \partial_t \eta_{v^\pm_\sigma}\, \mathrm{d}x \\ &\le \|\xi_{v^\pm_\sigma}\|_{0,\Omega} \|\partial_t \eta_{v^\pm_\sigma}\|_{0,\Omega} \\ &\le \|\xi_{v^\pm_\sigma}\|^2_{0,\Omega} + \|\partial_t \eta_{v^\pm_\sigma}\|^2_{0,\Omega} \\ &\le \|\xi_{v^\pm_\sigma}\|^2_{0,\Omega} + Ch^{2k+2} \|\partial_t v^\pm_\sigma\|^2_{L^\infty(0,T;H^{k+1}(\Omega))}.\end{aligned}$$

This leads to the estimate of Lemma 4.5 after multiplying the previous inequality by $\mathcal{A}_\sigma$ and summing σ (respectively β) over the bag (respectively branch) set $\mathfrak{L}^{\mathrm{NM}}$ (respectively $\{-,+\}$). □

4.3.1 *The transverse error term.* The transverse error term is controlled according to the following lemma.

LEMMA 4.6 Let $k \ge 1$ and $k' \ge 1$. Then there exists a constant C such that the transverse error term is bounded as

$$\sum_{\beta,\sigma} \sum_{K\in\mathcal{M}_h} \mathcal{A}_\sigma T^{\beta,\sigma}_{\perp K} \le C\Big(h^{2\min(k+1/2,k')} + \sum_{\beta,\sigma} \mathcal{A}_\sigma \|\xi_{v^\beta_\sigma}\|^2_{0,\Omega}\Big) - \kappa_\perp \sum_{\beta,\sigma} \mathcal{A}_\sigma \int_{\mathscr{E}_{h_\perp}} \mathcal{D}_{\perp,e} \left| [\![\xi_{v^\beta_\sigma}]\!]_\perp \cdot \nu_e \right|^2 \mathrm{d}\gamma,$$

where $0 < \kappa_\perp < 1/2$ is a purely numerical constant. The constant C depends on $\sum_\sigma \mathcal{A}_\sigma$, $\|\phi\|_{L^\infty(0,T;H^{k'+1}(\Omega))}$, $\|\phi\|_{L^\infty(0,T;W^{2,\infty}(\Omega))}$ and $\sum_{\beta,\sigma} \mathcal{A}_\sigma C\left(\|v^\pm_\sigma\|_{L^\infty(0,T;H^{k+1}(\Omega))}, \|v^\pm_\sigma\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}\right)$.

REMARK 4.7 The order $k + 1/2$ is sharp as we deal with a quite general situation. If we deal with more tricky projection-interpolation operators instead of the standard L^2-projection then we could recover optimal order $k + 1$. As examples, when the finite element is a cube we can use a tensorial product of one-dimensional Gauss–Radau projection, which was first introduced in Lesaint & Raviart (1974), and is now used repeatedly in the proof of the superconvergence property (Lesaint & Raviart, 1974; Cockburn *et al.*, 2001; Meng *et al.*, 2012). For simplicial finite elements in Cockburn *et al.* (2008, 2009), the authors have developed an L^2-projection operator with exact collocation at one of the boundary edges on each cell. For prismatic finite elements an idea may be to construct a tensorial product of a one-dimensional Gauss–Radau projection with the two-dimensional L^2-projection operator for simplicial finite elements developed in Cockburn *et al.* (2008). Actually, exact collocation at one of the boundary edges plus the orthogonality property for polynomials of degree up to $k-1$, used in combination with upwind fluxes, are key ingredients to obtain optimal order of convergence $k+1$ since the boundary error term vanishes (because of exact integration).

REMARK 4.8 The error estimate is optimal in terms of the space $\Phi_h^{k'}$. If we want to use the same degree of polynomial k for the electrical potential and the approximation of contours, while keeping the same order of approximation $k+1$ for the electrical potential and its perpendicular gradient (and thus obtaining an order $k+1/2$ for the contour approximations) then we could use a discretization space constructed as the tensorial product of mixed finite elements in the perpendicular direction such as the Raviart–Thomas–Nédélec (or edges) finite element of order k (with two-dimensional triangular or rectangular shape element) and discontinuous finite elements (e.g., orthogonal basis on each one-dimensional element) of degree k in the parallel direction.

Another point of view is to use the space Φ_h^{k+1} and thus we obtain an error estimate of order $k+1/2$ for the inequality of Lemma 4.6. On the one hand, resolution of the quasi-neutrality equation reduces to solving a series of independent (in parallel) two-dimensional elliptic (global) problems due to the natural decoupling between transverse (differential coupling) and longitudinal (algebraic coupling) directions. On the other hand, resolution of contour equations consists in solving a series of independent (in parallel) three-dimensional transport (local) problems. Therefore, in terms of computational cost it is worth using the space Φ_h^{k+1} for the electrical potential in combination with the space V_h^k for contours.

Proof. After performing an integration by parts on the first term of the right-hand side of (4.6) and using the exact incompressibility condition satisfied by v_E, we obtain the following decomposition

$$\sum_{K\in\mathcal{M}_h} T_{\perp K}^{\pm,\sigma} = T_1 + T_2 + T_3,$$

with the definitions

$$\begin{aligned}
T_1 = T_{11} + T_{12} = &-\sum_{K\in\mathcal{M}_h}\int_K v_E(\mathcal{J}_\perp\phi - \mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nabla_\perp v_\sigma^\pm \xi_{v_\sigma^\pm}\,\mathrm{d}x \\
&+\sum_{K\in\mathcal{M}_h}\int_{\partial K} v_E(\mathcal{J}_\perp\phi - \mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nu_{K_\perp} v_\sigma^\pm \xi_{v_\sigma^\pm}^\ell\,\mathrm{d}\gamma,
\end{aligned}$$

$$\begin{aligned}
T_2 &= \sum_{K\in\mathcal{M}_h}\int_K \frac{1}{2} v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nabla_\perp(\xi_{v_\sigma^\pm}^2)\,\mathrm{d}x - \sum_{K\in\mathcal{M}_h}\int_K v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nabla_\perp\xi_{v_\sigma^\pm}\eta_{v_\sigma^\pm}\,\mathrm{d}x \\
&= \sum_{K\in\mathcal{M}_h}\int_{\partial K} \frac{1}{2} v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nu_{K_\perp} {\xi_{v_\sigma^\pm}^\ell}^2\,\mathrm{d}\gamma - \sum_{K\in\mathcal{M}_h}\int_K v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nabla_\perp\xi_{v_\sigma^\pm}\eta_{v_\sigma^\pm}\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathscr{E}_{h_\perp}} \frac{1}{2} v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \cdot \left[\!\left[\xi^2_{v^\pm_\sigma} \right]\!\right]_\perp \mathrm{d}\gamma - \sum_{K \in \mathcal{M}_h} \int_K v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \cdot \nabla_\perp \xi_{v^\pm_\sigma} \eta_{v^\pm_\sigma} \, \mathrm{d}x \\
&= T_{21} + T_{22}
\end{aligned}
$$

and

$$
\begin{aligned}
T_3 = T_{31} + T_{32} = &\sum_{K \in \mathcal{M}_h} \int_{\partial K} v^\pm_\sigma v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h - \mathcal{J}_\perp \phi) \cdot \nu_{K_\perp} \xi^\ell_{v^\pm_\sigma} \, \mathrm{d}\gamma \\
&+ \sum_{K \in \mathcal{M}_h} \int_{\partial K} \left(\left\langle\!\left\langle v^\pm_{\sigma h} v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \right\rangle\!\right\rangle - v^\pm_\sigma v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \right) \cdot \nu_{K_\perp} \xi^\ell_{v^\pm_\sigma} \, \mathrm{d}\gamma.
\end{aligned}
$$

We first notice that $T_{31} + T_{12} = 0$. Let us look at the term T_{32}, which can be rewritten as

$$
\begin{aligned}
T_{32} &= -\int_{\mathscr{E}_{h_\perp}} \left\langle\!\!\left\langle e_{v^\pm_\sigma} v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \right\rangle\!\!\right\rangle \cdot \left[\!\left[\xi_{v^\pm_\sigma} \right]\!\right]_\perp \mathrm{d}\gamma = T_{321} + T_{322} \\
&= -\int_{\mathscr{E}_{h_\perp}} \left\langle\!\!\left\langle \xi_{v^\pm_\sigma} v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \right\rangle\!\!\right\rangle \cdot \left[\!\left[\xi_{v^\pm_\sigma} \right]\!\right]_\perp \mathrm{d}\gamma + \int_{\mathscr{E}_{h_\perp}} \left\langle\!\!\left\langle \eta_{v^\pm_\sigma} v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \right\rangle\!\!\right\rangle \cdot \left[\!\left[\xi_{v^\pm_\sigma} \right]\!\right]_\perp \mathrm{d}\gamma.
\end{aligned}
$$

Noting that

$$
\xi^\ell_{v^\pm_\sigma} = \{\!\{ \xi_{v^\pm_\sigma} \}\!\} + \frac{1}{2} \left[\!\left[\xi_{v^\pm_\sigma} \right]\!\right]_\perp \cdot \nu_{K_\perp}, \quad \left(\xi^\ell_{v^\pm_\sigma} \right)^2 = \{\!\{ \xi^2_{v^\pm_\sigma} \}\!\} + \{\!\{ \xi_{v^\pm_\sigma} \}\!\} \left[\!\left[\xi_{v^\pm_\sigma} \right]\!\right]_\perp \cdot \nu_{K_\perp},
$$

using the flux definition (3.10) and observing that

$$
\sum_{K \in \mathcal{M}_h} \sum_{e \in \partial K} \int_e \left(\left\langle\!\!\left\langle \xi_{v^\pm_\sigma} v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \right\rangle\!\!\right\rangle \{\!\{ \xi_{v^\pm_\sigma} \}\!\} - \frac{1}{2} v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \{\!\{ \xi^2_{v^\pm_\sigma} \}\!\} \right) \cdot \nu_{K_\perp} \mathrm{d}\gamma = 0,
$$

because of the continuity of the integrand across the edges, the summed term $T_{21} + T_{321}$ gives

$$
\begin{aligned}
T_{21} + T_{321} &= -\sum_{K \in \mathcal{M}_h} \sum_{e \in \partial K} \int_e \left(\left\langle\!\!\left\langle \xi_{v^\pm_\sigma} v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \right\rangle\!\!\right\rangle \xi^\ell_{v^\pm_\sigma} - \frac{1}{2} v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \left(\xi^\ell_{v^\pm_\sigma} \right)^2 \right) \cdot \nu_{K_\perp} \mathrm{d}\gamma \\
&= -\sum_{K \in \mathcal{M}_h} \sum_{e \in \partial K} \int_e \Bigg\{ \left\langle\!\!\left\langle \xi_{v^\pm_\sigma} v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \right\rangle\!\!\right\rangle \cdot \nu_{K_\perp} \left(\{\!\{ \xi_{v^\pm_\sigma} \}\!\} + \frac{1}{2} \left[\!\left[\xi_{v^\pm_\sigma} \right]\!\right]_\perp \cdot \nu_{K_\perp} \right) \\
&\qquad - \frac{1}{2} v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \cdot \nu_{K_\perp} \left(\{\!\{ \xi^2_{v^\pm_\sigma} \}\!\} + \{\!\{ \xi_{v^\pm_\sigma} \}\!\} \left[\!\left[\xi_{v^\pm_\sigma} \right]\!\right]_\perp \cdot \nu_{K_\perp} \right) \Bigg\} \mathrm{d}\gamma \\
&= -\sum_{K \in \mathcal{M}_h} \sum_{e \in \partial K} \frac{1}{2} \int_e \left[\!\left[\xi_{v^\pm_\sigma} \right]\!\right]_\perp \cdot \nu_{K_\perp} \left(\left\langle\!\!\left\langle \xi_{v^\pm_\sigma} v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \right\rangle\!\!\right\rangle - v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \{\!\{ \xi_{v^\pm_\sigma} \}\!\} \right) \cdot \nu_{K_\perp} \mathrm{d}\gamma \\
&= -\int_{\mathscr{E}_{h_\perp}} \frac{\mathcal{D}_{\perp,e}}{2} \left| \left[\!\left[\xi_{v^\beta_\sigma} \right]\!\right]_\perp \cdot \nu_{K_\perp} \right|^2 \mathrm{d}\gamma = T_\perp.
\end{aligned} \tag{4.10}
$$

Therefore, we obtain $T_1 + T_2 + T_3 = T_{11} + T_{22} + T_{322} + T_\perp$. Let us now estimate T_1. By (4.1), using stability and high-order approximation properties for the projection-interpolation operators (Section 3.1.3), stability properties of the gyroaverage operator (Section 3.1.4) and the Young inequality, we obtain

$$\begin{aligned}
T_{11} &= \int_\Omega v_E(\mathcal{J}_\perp\phi - \mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nabla_\perp v_\sigma^\pm \xi_{v_\sigma^\pm}\,\mathrm{d}x \\
&\le |v_\sigma^\pm|_{1,\infty,\Omega}\{\|v_E([I-\mathcal{R}_h]\mathcal{J}_\perp\phi)\|_{0,\Omega} + \|v_E(\mathcal{R}_h\mathcal{J}_\perp[\phi-\phi_h])\|_{0,\Omega}\}\|\xi_{v_\sigma^\pm}\|_{0,\Omega} \\
&\le C_\sigma|v_\sigma^\pm|_{1,\infty,\Omega}\{h^{k'}\|\phi\|_{k'+1,\Omega} + \|\partial_\perp(\phi-\phi_h)\|_{0,\Omega}\}\|\xi_{v_\sigma^\pm}\|_{0,\Omega} \\
&\le C_\sigma h^{2\min(k+1,k')} + C_\sigma\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v_\sigma^\beta - v_{\sigma h}^\beta\right\|_{0,\Omega}^2 + \|\xi_{v_\sigma^\pm}\|_{0,\Omega}^2 \\
&\le C_\sigma h^{2\min(k+1,k')} + C_\sigma\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|\xi_{v_\sigma^\pm}\right\|_{0,\Omega}^2 + \|\xi_{v_\sigma^\pm}\|_{0,\Omega}^2.
\end{aligned}\tag{4.11}$$

From the definition of Π_h and using the decomposition for the term T_{22},

$$\begin{aligned}
T_{22} = T_{221} + T_{222} + T_{223} = &-\int_\Omega v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h - \mathcal{J}_\perp\phi)\cdot\nabla_\perp\xi_{v_\sigma^\pm}\eta_{v_\sigma^\pm}\,\mathrm{d}x \\
&-\int_\Omega [I-\pi_h]v_E(\mathcal{J}_\perp\phi)\cdot\nabla_\perp\xi_{v_\sigma^\pm}\eta_{v_\sigma^\pm}\,\mathrm{d}x - \int_\Omega \pi_h v_E(\mathcal{J}_\perp\phi)\cdot\nabla_\perp\xi_{v_\sigma^\pm}\eta_{v_\sigma^\pm}\,\mathrm{d}x,
\end{aligned}$$

we first get $T_{223} = 0$. Next, using inverse inequalities, stability and high-order approximation properties for the projection-interpolation operators (Section 3.1.3), stability properties of the gyroaverage operator (Section 3.1.4), estimate (4.2) and the Young inequality, we obtain

$$\begin{aligned}
T_{221} &\le (h|\phi|_{2,\infty,\Omega} + \|\partial_\perp(\phi-\phi_h)\|_{0,\infty,\Omega})|\xi_{v_\sigma^\pm}|_{1,\Omega}\|\eta_{v_\sigma^\pm}\|_{0,\Omega} \\
&\le Ch^{k+1}\|v_\sigma^\pm\|_{k+1,\Omega}\|\xi_{v_\sigma^\pm}\|_{0,\Omega}C_\infty(\|\phi\|_{2,\infty,\Omega})\left\{1 + h^{-3/2}|\ln h|\left(\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v_\sigma^\beta - v_{\sigma h}^\beta\right\|_{0,\Omega}^2\right)^{1/2}\right\} \\
&\le C_\sigma h^{2k+2} + C_\sigma h^{2k-1}|\ln h|^2\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v_\sigma^\beta - v_{\sigma h}^\beta\right\|_{0,\Omega}^2 + \|\xi_{v_\sigma^\pm}\|_{0,\Omega}^2 \\
&\le C_\sigma h^{2k+2} + C_\sigma h^{4k+1}|\ln h|^2 + C_\sigma h^{2k-1}|\ln h|^2\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|\xi_{v_\sigma^\pm}\right\|_{0,\Omega}^2 + \|\xi_{v_\sigma^\pm}\|_{0,\Omega}^2 \\
&\le C_\sigma h^{2k+2} + C_\sigma\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|\xi_{v_\sigma^\pm}\right\|_{0,\Omega}^2 + \|\xi_{v_\sigma^\pm}\|_{0,\Omega}^2
\end{aligned}\tag{4.12}$$

and

$$\begin{aligned}
T_{222} &\le Ch|v_E(\phi)|_{1,\infty,\Omega}|\xi_{v_\sigma^\pm}|_{1,\Omega}\|\eta_{v_\sigma^\pm}\|_{0,\Omega} \\
&\le Ch^{k+1}|\phi|_{2,\infty,\Omega}\|v_\sigma^\pm\|_{k+1,\Omega}\|\xi_{v_\sigma^\pm}\|_{0,\Omega} \\
&\le C_\sigma h^{2k+2} + \|\xi_{v_\sigma^\pm}\|_{0,\Omega}^2.
\end{aligned}\tag{4.13}$$

It remains to bound T_{322}. By (3.10), using the Cauchy–Schwarz and the Young inequalities, stability properties for the projection-interpolation (Section 3.1.3) and gyroaverage (Section 3.1.4) operators and (4.2), we obtain

$$
\begin{aligned}
T_{322} &= \int_{\mathscr{E}_{h_\perp}} \{\!\{\eta_{v_\sigma^\pm}\}\!\} v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot [\![\xi_{v_\sigma^\pm}]\!]_\perp \,\mathrm{d}\gamma + \int_{\mathscr{E}_{h_\perp}} \frac{\mathcal{D}_{\perp,e}}{2} [\![\eta_{v_\sigma^\pm}]\!]_\perp \cdot [\![\xi_{v_\sigma^\pm}]\!]_\perp \,\mathrm{d}\gamma \\
&\le \sum_{e\in\mathscr{E}_{h_\perp}} \left\| |v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot \nu_e|^{1/2} \left| [\![\xi_{v_\sigma^\pm}]\!]_\perp \cdot \nu_e \right| \right\|_{0,e} \left\| |v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot \nu_e|^{1/2} \left| \{\!\{\eta_{v_\sigma^\pm}\}\!\} \right| \right\|_{0,e} \\
&\quad + \frac{1}{2}\sum_{e\in\mathscr{E}_{h_\perp}} \left\| \mathcal{D}_{\perp,e}^{1/2} \left| [\![\xi_{v_\sigma^\pm}]\!]_\perp \cdot \nu_e \right| \right\|_{0,e} \left\| \mathcal{D}_{\perp,e}^{1/2} \left| [\![\eta_{v_\sigma^\pm}]\!]_\perp \cdot \nu_e \right| \right\|_{0,e} \\
&\le \varepsilon \int_{\mathscr{E}_{h_\perp}} \mathcal{D}_{\perp,e} \left| [\![\eta_{v_\sigma^\pm}]\!]_\perp \cdot \nu_e \right|^2 \mathrm{d}\gamma \\
&\quad + C(\varepsilon)\|v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nu_e\|_{0,\infty,\Omega} \sum_{K\in\mathcal{M}_h}\sum_{e\in\partial K} \left(\left\| [\![\eta_{v_\sigma^\pm}]\!]_\perp \cdot \nu_e \right\|_{0,e}^2 + \left\| \{\!\{\eta_{v_\sigma^\pm}\}\!\} \right\|_{0,e}^2 \right) \\
&\le \varepsilon \int_{\mathscr{E}_{h_\perp}} \mathcal{D}_{\perp,e} \left| [\![\eta_{v_\sigma^\pm}]\!]_\perp \cdot \nu_e \right|^2 \mathrm{d}\gamma + Ch^{-1}\Big(\|v_E(\mathcal{R}_h\mathcal{J}_\perp\phi)\|_{0,\infty,\Omega} \\
&\quad + \|v_E(\mathcal{R}_h\mathcal{J}_\perp[\phi_h-\phi])\|_{0,\infty,\Omega}\Big) \left\|\eta_{v_\sigma^\pm}\right\|_{0,\Omega}^2 \\
&\le \varepsilon \int_{\mathscr{E}_{h_\perp}} \mathcal{D}_{\perp,e} \left| [\![\eta_{v_\sigma^\pm}]\!]_\perp \cdot \nu_e \right|^2 \mathrm{d}\gamma + Ch^{2k+1}\|v_\sigma^\pm\|_{k+1,\Omega}^2 \left(|\phi|_{1,\infty,\Omega} + \|\partial_\perp(\phi-\phi_h)\|_{0,\infty,\Omega} \right) \\
&\le \varepsilon \int_{\mathscr{E}_{h_\perp}} \mathcal{D}_{\perp,e} \left| [\![\eta_{v_\sigma^\pm}]\!]_\perp \cdot \nu_e \right|^2 \mathrm{d}\gamma + C_\sigma h^{2k+1} + C_\sigma h^{2k+1/2}|\ln h| \Big(\sum_{\beta,\sigma} \mathcal{A}_\sigma \left\| v_\sigma^\beta - v_{\sigma h}^\beta \right\|_{0,\Omega}^2 \Big)^{1/2} \\
&\le \varepsilon \int_{\mathscr{E}_{h_\perp}} \mathcal{D}_{\perp,e} \left| [\![\eta_{v_\sigma^\pm}]\!]_\perp \cdot \nu_e \right|^2 \mathrm{d}\gamma + C_\sigma h^{2k+1} + C_\sigma h^{4k+1}|\ln h|^2 + C_\sigma \sum_{\beta,\sigma} \mathcal{A}_\sigma \left\| v_\sigma^\beta - v_{\sigma h}^\beta \right\|_{0,\Omega}^2 \\
&\le \varepsilon \int_{\mathscr{E}_{h_\perp}} \mathcal{D}_{\perp,e} \left| [\![\eta_{v_\sigma^\pm}]\!]_\perp \cdot \nu_e \right|^2 \mathrm{d}\gamma + C_\sigma h^{2k+1} + C_\sigma \sum_{\beta,\sigma} \mathcal{A}_\sigma \left\| \xi_{v_\sigma^\pm} \right\|_{0,\Omega}^2 .
\end{aligned} \tag{4.14}
$$

Gathering estimates of the terms $T_\perp$, T_{11}, T_{221}, T_{222} and T_{322}, which are given, respectively, by (4.10), (4.11), (4.12), (4.13) and (4.14), multiplying them by $\mathcal{A}_\sigma$ and summing σ (respectively β) over the bag (respectively branch) set $\mathfrak{L}^{\mathrm{NM}}$ (respectively $\{-,+\}$), we obtain the estimate of Lemma 4.6. □

4.3.2 *The longitudinal error term arising from the electrical potential.*

LEMMA 4.9 Let $k \ge 0$ and $k' \ge 1$. Then there exists a constant C such that the parallel error term coming from the electrical potential is bounded as

$$\sum_{\beta,\sigma}\sum_{K\in\mathcal{M}_h} \mathcal{A}_\sigma T_{\parallel K,\phi}^{\beta,\sigma} \le C\Big(h^{2\min(k,k')} + \sum_{\beta,\sigma} \mathcal{A}_\sigma \|\xi_{v_\sigma^\beta}\|_{0,\Omega}^2 \Big),$$

where the constant C depends on $\sum_\sigma \mathcal{A}_\sigma$, $\|\phi\|_{L^\infty\left(0,T;H^{k'+1}(\Omega)\right)}$, $\|\phi\|_{L^\infty\left(0,T;W^{2,\infty}(\Omega)\right)}$ and $\sum_{\beta,\sigma}\mathcal{A}_\sigma$ $C(\|v_\sigma^\pm\|_{L^\infty(0,T;H^{k+1}(\Omega))}, \|v_\sigma^\pm\|_{L^\infty(0,T;W^{1,\infty}(\Omega))})$.

Remark 4.10 To obtain an estimate of order $\min(k,k')+1$ instead of $\min(k,k')$ two ingredients are required. First, we need to estimate the source-term errors $\bar{\rho}-\bar{\rho}_h$ and $\tilde{\rho}-\tilde{\rho}_h$ in the H^{-1}-norm (see the proof of Proposition 4.3 in Appendix A.1). This leads to an additional multiplicative factor h with respect to the L^2-norm. This procedure is made possible since we have a potential gain of derivatives (in the perpendicular direction) for elliptic problems. Second, we need to use a superconvergent mixed (Cockburn *et al.*, 2008) or discontinuous Galerkin (Cockburn *et al.*, 2009) finite element method with special postprocessing projections to get an approximation of the electrical potential converging in the L^2-norm with order $k'+2$.

Proof. Using definition (4.9) of the error term $T^{\pm,\sigma}_{\parallel K,\phi}$, we obtain

$$\sum_{K\in\mathcal{M}_h} T^{\pm,\sigma}_{\parallel K,\phi} = \int_\Omega (\mathcal{J}_\perp\phi - \mathcal{R}_h\mathcal{J}_\perp\phi_h)\,\partial_\parallel \xi_{v_\sigma^\pm}\,\mathrm{d}x - \int_{\mathcal{E}_{h_\parallel}} (\mathcal{J}_\perp\phi - \langle\!\langle \mathcal{R}_h\mathcal{J}_\perp\phi_h\rangle\!\rangle)\,[\![\xi_{v_\sigma^\pm}]\!]_\parallel\,\mathrm{d}\gamma = T_1+T_2. \quad (4.15)$$

Let us start with the first term T_1 of (4.15). Using the Young inequality, properties (stability, high-order approximation and inverse inequalities) for the projection-interpolation operators (Section 3.1.3), stabililty properties of the gyroaverage operator (Section 3.1.4) and (4.1), we obtain

$$\begin{aligned} T_1 &\le h^{-2}\|\mathcal{J}_\perp\phi - \mathcal{R}_h\mathcal{J}_\perp\phi_h\|^2_{0,\Omega} + \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} \\ &\le h^{-2}\|(I-\mathcal{R}_h)\mathcal{J}_\perp\phi\|^2_{0,\Omega} + h^{-2}\|\mathcal{R}_h\mathcal{J}_\perp(\phi-\phi_h)\|^2_{0,\Omega} + \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} \\ &\le Ch^{2k'}|\phi|_{k'+1,\Omega} + Ch^{-2}\|\phi-\phi_h\|^2_{0,\Omega} + \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} \\ &\le C_\sigma h^{2\min(k,k')} + C_\sigma \sum_{\beta,\sigma}\mathcal{A}_\sigma \|v_\sigma^\pm - v_{\sigma h}^\pm\|^2_{0,\Omega} + \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} \\ &\le C_\sigma h^{2\min(k,k')} + C_\sigma \sum_{\beta,\sigma}\mathcal{A}_\sigma \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} + \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega}. \end{aligned} \quad (4.16)$$

We then deal with the term T_2 of (4.15). Using the flux definition (3.12), the Young inequality, properties (stability, high-order approximation and inverse inequalities) for the projection-interpolation operators (Section 3.1.3), stability properties of the gyroaverage operator (Section 3.1.4) and (4.1), we obtain

$$\begin{aligned} T_2 &= \sum_{K\in\mathcal{M}_h}\sum_{e\in\partial K}\int_e (\mathcal{J}_\perp\phi - \langle\!\langle \mathcal{R}_h\mathcal{J}_\perp\phi_h\rangle\!\rangle)\nu_{K_\parallel}\xi^\ell_{v_\sigma^\pm}\,\mathrm{d}\gamma \\ &\le \sum_{K\in\mathcal{M}_h}\sum_{e\in\partial K}\left\{Ch^{-1}\|\{\!\{\mathcal{J}_\perp\phi - \mathcal{R}_h\mathcal{J}_\perp\phi_h\}\!\}_\alpha\|^2_{0,e} + h\|\xi_{v_\sigma^\pm}\|^2_{0,e}\right\} \\ &\le \sum_{K\in\mathcal{M}_h}\sum_{e\in\partial K}\left\{Ch^{-2}\|\mathcal{J}_\perp\phi - \mathcal{R}_h\mathcal{J}_\perp\phi_h\|^2_{0,K} + \|\xi_{v_\sigma^\pm}\|^2_{0,K}\right\} \end{aligned}$$

$$
\begin{aligned}
&\le h^{-2}\|(I-\mathcal{R}_h)\mathcal{J}_\perp\phi\|_{0,\Omega}^2 + h^{-2}\|\mathcal{R}_h\mathcal{J}_\perp(\phi-\phi_h)\|_{0,\Omega}^2 + \|\xi_{v_\sigma^\pm}\|_{0,\Omega}^2 \\
&\le Ch^{2k'}|\phi|_{k'+1,\Omega} + Ch^{-2}\|\phi-\phi_h\|_{0,\Omega}^2 + \|\xi_{v_\sigma^\pm}\|_{0,\Omega}^2 \\
&\le C_\sigma h^{2\min(k,k')} + C_\sigma \sum_{\beta,\sigma}\mathcal{A}_\sigma\|v_\sigma^\pm - v_{\sigma h}^\pm\|_{0,\Omega}^2 + \|\xi_{v_\sigma^\pm}\|_{0,\Omega}^2 \\
&\le C_\sigma h^{2\min(k,k')} + C_\sigma \sum_{\beta,\sigma}\mathcal{A}_\sigma\|\xi_{v_\sigma^\pm}\|_{0,\Omega}^2 + \|\xi_{v_\sigma^\pm}\|_{0,\Omega}^2 .
\end{aligned} \tag{4.17}
$$

Gathering estimates of the terms T_1 and T_2, which are given, respectively, by (4.16) and (4.17), multiplying them by $\mathcal{A}_\sigma$ and summing σ (respectively β) over the bag (respectively branch) set $\mathfrak{L}^{\mathrm{NM}}$ (respectively $\{-,+\}$), we obtain the estimate of Lemma 4.9. □

4.3.3 *The longitudinal error term arising from contours*

LEMMA 4.11 Let $k \ge 1/2$. Then there exists a constant C such that the parallel error term coming from contours is bounded as

$$
\begin{aligned}
&\sum_{\beta,\sigma}\sum_{K\in\mathcal{M}_h}\mathcal{A}_\sigma T_{\parallel K,v}^{\beta,\sigma} \\
&\quad\le C\left(h^{2m} + \sum_{\beta,\sigma}\mathcal{A}_\sigma\left(\|\xi_{v_\sigma^\beta}\|_{0,\Omega}^2 + h^{-5/2}\|\xi_{v_\sigma^\beta}\|_{0,\Omega}^3\right)\right) - \kappa_\parallel\sum_{\beta,\sigma}\mathcal{A}_\sigma\int_{\mathcal{E}_{h_\parallel}}\mathcal{D}_{\parallel\sigma,e}^\beta\left|\left[\!\left[\xi_{v_\sigma^\beta}\right]\!\right]_\parallel\cdot\nu_e\right|^2\mathrm{d}\gamma,
\end{aligned}
$$

with $m = k$ if we use the global Lax–Friedrich flux for the parallel flux (3.11) and $m = k + 1/2$ in the other cases.

The purely numerical constant $\kappa_\parallel$ is such that $0 < \kappa_\parallel < 1/2$, while the constant C depends on $\sum_\sigma \mathcal{A}_\sigma$, $\|\phi\|_{L^\infty\left(0,T;H^{k'+1}(\Omega)\right)}$, $\|\phi\|_{L^\infty(0,T;W^{2,\infty}(\Omega))}$ and $\sum_{\beta,\sigma}\mathcal{A}_\sigma C\left(\|v_\sigma^\pm\|_{L^\infty\left(0,T;H^{k+1}(\Omega)\right)},\ \|v_\sigma^\pm\|_{L^\infty\left(0,T;W^{1,\infty}(\Omega)\right)}\right)$.

Proof. From the flux definition (3.11) and definition (4.8) of the error term $T_{\parallel K,v}^{\pm,\sigma}$, after using two integration by parts and some little algebra we find

$$
\sum_{K\in\mathcal{M}_h} T_{\parallel K,v}^{\pm,\sigma} = T_1 + T_2,
$$

where

$$
\begin{aligned}
T_1 &= -\int_\Omega \frac{1}{2}\xi_{v_\sigma^\pm}^2\partial_\parallel v_\sigma^\pm\,\mathrm{d}x - \int_\Omega v_\sigma^\pm\eta_{v_\sigma^\pm}\partial_\parallel\xi_{v_\sigma^\pm}\,\mathrm{d}x - \int_\Omega\frac{1}{2}\eta_{v_\sigma^\pm}^2\partial_\parallel\xi_{v_\sigma^\pm}\,\mathrm{d}x + \int_\Omega \eta_{v_\sigma^\pm}\xi_{v_\sigma^\pm}\partial_\parallel\xi_{v_\sigma^\pm}\,\mathrm{d}x \\
&= T_{11} + T_{12} + T_{13} + T_{14}
\end{aligned}
$$

and

$$
\begin{aligned}
T_2 = &-\int_{\mathcal{E}_{h_\parallel}} \frac{1}{2}\mathcal{D}^{\pm}_{\parallel\sigma,e}\left|[\![\xi_{v_\sigma^\pm}]\!]_\parallel \cdot \nu_e\right|^2 \mathrm{d}\gamma + \int_{\mathcal{E}_{h_\parallel}} \frac{1}{2}\mathcal{D}^{\pm}_{\parallel\sigma}[\![\eta_{v_\sigma^\pm}]\!]_\parallel [\![\xi_{v_\sigma^\pm}]\!]_\parallel \,\mathrm{d}\gamma + \int_{\mathcal{E}_{h_\parallel}} \{\!\{\eta_{v_\sigma^\pm}\}\!\}[\![\xi_{v_\sigma^\pm}]\!]_\parallel v_\sigma^\pm \,\mathrm{d}\gamma \\
&+ \int_{\mathcal{E}_{h_\parallel}} \frac{1}{2}[\![\xi_{v_\sigma^\pm}]\!]_\parallel \{\!\{\eta^2_{v_\sigma^\pm}\}\!\}\,\mathrm{d}\gamma - \int_{\mathcal{E}_{h_\parallel}} [\![\xi_{v_\sigma^\pm}]\!]_\parallel \{\!\{\xi_{v_\sigma^\pm}\eta_{v_\sigma^\pm}\}\!\}\,\mathrm{d}\gamma + \frac{1}{12}\int_{\mathcal{E}_{h_\parallel}} [\![\xi_{v_\sigma^\pm}]\!]^3_\parallel \,\mathrm{d}\gamma \\
= &\, T_\parallel + T_{21} + T_{22} + T_{23} + T_{24} + T_{25}.
\end{aligned}
$$

Let us start with T_{11} bounded as

$$
T_{11} \le |v_\sigma^\pm|_{1,\infty,\Omega}\|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} \le C_\sigma\|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} \le C\|\xi_{v_\sigma^\pm}\|^2_{0,\Omega}. \tag{4.18}
$$

The term T_{12} can be decomposed as

$$
T_{12} = T_{121} + T_{122} = -\int_\Omega (v_\sigma^\pm - \pi_h v_\sigma^\pm)\eta_{v_\sigma^\pm}\partial_\parallel \xi_{v_\sigma^\pm}\,\mathrm{d}x - \int_\Omega \pi_h v_\sigma^\pm \eta_{v_\sigma^\pm}\partial_\parallel\xi_{v_\sigma^\pm}\,\mathrm{d}x.
$$

We have $T_{122} = 0$ by definition of the L^2-projection operator Π_h and π_h. Using the inverse inequalities and high-order approximation properties for the projection-interpolation operators (Section 3.1.3) and the Young inequality, we find

$$
\begin{aligned}
T_{121} \le \|v_\sigma^\pm - \pi_h v_\sigma^\pm\|_{0,\infty,\Omega}\|\eta_{v_\sigma^\pm}\|_{0,\Omega}\|\partial_\parallel\xi_{v_\sigma^\pm}\|_{0,\Omega} &\le C|v_\sigma^\pm|_{1,\infty,\Omega}\|\eta_{v_\sigma^\pm}\|_{0,\Omega}\|\xi_{v_\sigma^\pm}\|_{0,\Omega} \\
&\le C_\sigma h^{2k+2} + \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega}.
\end{aligned} \tag{4.19}
$$

Using inverse inequalities and high-order approximation properties for the projection-interpolation operators (Section 3.1.3), the Cauchy–Schwarz estimate and the Young inequality, we obtain for the term T_{13},

$$
\begin{aligned}
T_{13} &\le \|\partial_\parallel\xi_{v_\sigma^\pm}\|_{0,\infty,\Omega}\|\eta_{v_\sigma^\pm}\|^2_{0,\Omega} \\
&\le Ch^{2k+1}\|\xi_{v_\sigma^\pm}\|_{0,\infty,\Omega}\|v_\sigma^\pm\|^2_{0,\Omega} \\
&\le C_\sigma h^{2k-1/2}\|\xi_{v_\sigma^\pm}\|_{0,\Omega} \\
&\le C_\sigma h^{3k+1/2} + h^{-5/2}\|\xi_{v_\sigma^\pm}\|^3_{0,\Omega} \\
&\le C_\sigma h^{2k+1} + h^{-5/2}\|\xi_{v_\sigma^\pm}\|^3_{0,\Omega},
\end{aligned} \tag{4.20}
$$

with $k \ge 1/2$. In the same way, we get for the term T_{14},

$$
\begin{aligned}
T_{14} &\le \|\xi_{v_\sigma^\pm}\|_{0,4,\Omega}\|\partial_\parallel\xi_{v_\sigma^\pm}\|_{0,4,\Omega}\|\eta_{v_\sigma^\pm}\|_{0,\Omega} \\
&\le Ch^k\|\xi_{v_\sigma^\pm}\|^2_{0,4,\Omega}\|v_\sigma^\pm\|_{0,\Omega} \\
&\le C_\sigma h^{k-3/2}\|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} \\
&\le C_\sigma h^{3k+1/2} + h^{-5/2}\|\xi_{v_\sigma^\pm}\|^3_{0,\Omega} \\
&\le C_\sigma h^{2k+1} + h^{-5/2}\|\xi_{v_\sigma^\pm}\|^3_{0,\Omega},
\end{aligned} \tag{4.21}
$$

with $k \geq 1/2$. We then deal with the term T_{21}. Using inverse inequalities, stability and high-order approximation properties for the projection-interpolation operators (Section 3.1.3), the Cauchy–Schwarz estimate and the Young inequality, we find

$$
\begin{aligned}
T_{21} &\leq \frac{1}{2} \left\| \mathcal{D}^{\pm\ 1/2}_{\parallel\sigma} \left| [\![\eta_{v^{\pm}_{\sigma}}]\!]_{\parallel} \cdot \nu_e \right| \right\|_{0,\mathcal{E}_{h_\parallel}} \left\| \mathcal{D}^{\pm\ 1/2}_{\parallel\sigma} \left| [\![\xi_{v^{\pm}_{\sigma}}]\!]_{\parallel} \cdot \nu_e \right| \right\|_{0,\mathcal{E}_{h_\parallel}} \\
&\leq C(\varepsilon) \int_{\mathcal{E}_{h_\parallel}} \mathcal{D}^{\pm}_{\parallel\sigma,e} \left| [\![\eta_{v^{\pm}_{\sigma}}]\!]_{\parallel} \cdot \nu_e \right|^2 \mathrm{d}\gamma + \frac{\varepsilon}{8} \int_{\mathcal{E}_{h_\parallel}} \mathcal{D}^{\pm}_{\parallel\sigma,e} \left| [\![\xi_{v^{\pm}_{\sigma}}]\!]_{\parallel} \cdot \nu_e \right|^2 \mathrm{d}\gamma = T_{211} + T_{212} \\
&\leq C h^{-1} \| \mathcal{D}^{\pm}_{\parallel\sigma} \|_{0,\infty,\Omega} \| \eta_{v^{\pm}_{\sigma}} \|^2_{0,\Omega} + T_{212} \\
&\leq C_\sigma h^{2k+1} \| v^{\pm}_{\sigma h} \|_{0,\infty,\Omega} + T_{212} \\
&\leq C_\sigma h^{2k+1} (\| \xi_{v^{\pm}_{\sigma}} \|_{0,\infty,\Omega} + \| \Pi_h v^{\pm}_{\sigma} \|_{0,\infty,\Omega}) + T_{212} \\
&\leq C_\sigma h^{2k+1} \left(1 + h^{-3/2} \| \xi_{v^{\pm}_{\sigma}} \|_{0,\Omega} \right) + T_{212} \\
&\leq C_\sigma (h^{2k+1} + h^{3k+1/2}) + h^{-5/2} \| \xi_{v^{\pm}_{\sigma}} \|^3_{0,\Omega} + T_{212} \\
&\leq C_\sigma h^{2k+1} + h^{-5/2} \| \xi_{v^{\pm}_{\sigma}} \|^3_{0,\Omega} + \frac{\varepsilon}{8} \int_{\mathcal{E}_{h_\parallel}} \mathcal{D}^{\pm}_{\parallel\sigma,e} \left| [\![\xi_{v^{\pm}_{\sigma}}]\!]_{\parallel} \cdot \nu_e \right|^2 \mathrm{d}\gamma. \qquad (4.22)
\end{aligned}
$$

Let us now estimate the term T_{22} that we decompose as

$$
T_{22} = T_{221} + T_{222} = \pm \int_{\mathcal{E}_{h_\parallel}} \mathcal{D}^{\pm}_{\parallel\sigma} \{\!\{ \eta_{v^{\pm}_{\sigma}} \}\!\} [\![\xi_{v^{\pm}_{\sigma}}]\!]_{\parallel} \, \mathrm{d}\gamma + \int_{\mathcal{E}_{h_\parallel}} (v^{\pm}_{\sigma} \mp \mathcal{D}^{\pm}_{\parallel\sigma}) \{\!\{ \eta_{v^{\pm}_{\sigma}} \}\!\} [\![\xi_{v^{\pm}_{\sigma}}]\!]_{\parallel} \, \mathrm{d}\gamma.
$$

The term T_{221} can be bounded in the same way as the term T_{21}. If the parallel flux (3.11) is either the upwind flux or the local Lax–Friedrich flux with $S = \{x\}$ then we obtain the following bound for T_{222}. Using inverse inequalities and high-order approximation properties for the projection-interpolation operators (Section 3.1.3) and the Young inequality we get

$$
\begin{aligned}
T_{222} &\leq \sum_{K \in \mathcal{M}_h} \sum_{e \in \partial K} \| \eta_{v^{\pm}_{\sigma}} \|_{0,\infty,K} \| v^{\pm}_{\sigma} - v^{\pm}_{\sigma h} \|_{0,e} \| \xi_{v^{\pm}_{\sigma}} \|_{0,e} \\
&\leq C \sum_{K \in \mathcal{M}_h} | v^{\pm}_{\sigma} |_{1,\infty,K} \| v^{\pm}_{\sigma} - v^{\pm}_{\sigma h} \|_{0,K} \| \xi_{v^{\pm}_{\sigma}} \|_{0,K} \\
&\leq C_\sigma \| v^{\pm}_{\sigma} - v^{\pm}_{\sigma h} \|_{0,\Omega} \| \xi_{v^{\pm}_{\sigma}} \|_{0,\Omega} \\
&\leq C_\sigma \| \eta_{v^{\pm}_{\sigma}} \|^2_{0,\Omega} + C \| \xi_{v^{\pm}_{\sigma}} \|^2_{0,\Omega} \\
&\leq C_\sigma h^{2k+2} + C \| \xi_{v^{\pm}_{\sigma}} \|^2_{0,\Omega}. \qquad (4.23)
\end{aligned}
$$

Now, if the parallel flux (3.11) is the local Lax–Friedrich flux with $S = e,\ \partial K,\ \partial K \cup \partial K'$ $(K \cap K' \neq \emptyset)$ then we proceed as follows. Let us define

$$
x^{*\pm}_m = \arg\min_{x \in S} \left(v^{\pm,\ell}_{\sigma h}, v^{\pm,r}_{\sigma h} \right) \quad \text{and} \quad x^{*\pm}_M = \arg\max_{x \in S} \left(v^{\pm,\ell}_{\sigma h}, v^{\pm,r}_{\sigma h} \right).
$$

Then, with $s \in \{m, M\}$ and $\beta \in \{-, +\}$, we have the decomposition,

$$T_{222} = \sum_{K\in\mathcal{M}_h}\sum_{e\in\partial K}\int_e \left(v_\sigma^\pm(x) - v_\sigma^\pm(x_s^{*\beta})\right) \{\!\{\eta_{v_\sigma^\pm}\}\!\}\xi^\ell_{v_\sigma^\pm}\nu_{K_\parallel}\,\mathrm{d}\gamma$$
$$+ \sum_{K\in\mathcal{M}_h}\sum_{e\in\partial K}\int_e \left(v_\sigma^\pm(x_s^{*\beta}) - v_{\sigma h}^\pm(x_s^{*\beta})\right) \{\!\{\eta_{v_\sigma^\pm}\}\!\}\xi^\ell_{v_\sigma^\pm}\nu_{K_\parallel}\,\mathrm{d}\gamma = T_{2221} + T_{2222}.$$

The term T_{2221} is bounded as

$$\begin{aligned} T_{2221} &\le Ch|v_\sigma^\pm|_{1,\infty,\Omega}\sum_{K\in\mathcal{M}_h}\sum_{e\in\partial K}\left\|\{\!\{\eta_{v_\sigma^\pm}\}\!\}\right\|_{0,e}\|\xi_{v_\sigma^\pm}\|_{0,e} \\ &\le C_\sigma\sum_{K\in\mathcal{M}_h}\|\eta_{v_\sigma^\pm}\|_{0,K}\|\xi_{v_\sigma^\pm}\|_{0,K} \\ &\le C_\sigma h^{2k+2} + \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega}, \end{aligned} \tag{4.24}$$

whereas the term T_{2222} is bounded as

$$\begin{aligned} T_{2222} &\le \|v_\sigma^\pm - v_{\sigma h}^\pm\|_{0,\infty,\Omega}\sum_{K\in\mathcal{M}_h}\sum_{e\in\partial K}\left\|\{\!\{\eta_{v_\sigma^\pm}\}\!\}\right\|_{0,e}\|\xi_{v_\sigma^\pm}\|_{0,e} \\ &\le Ch^{-1}\left(\|\eta_{v_\sigma^\pm}\|_{0,\infty,\Omega} + \|\xi_{v_\sigma^\pm}\|_{0,\infty,\Omega}\right)\|\eta_{v_\sigma^\pm}\|_{0,\Omega}\|\xi_{v_\sigma^\pm}\|_{0,\Omega} \\ &\le Ch^{-5/2}\|\eta_{v_\sigma^\pm}\|_{0,\Omega}\|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} + C|v_\sigma^\pm|_{1,\infty,\Omega}\|\eta_{v_\sigma^\pm}\|_{0,\Omega}\|\xi_{v_\sigma^\pm}\|_{0,\Omega} \\ &\le C_\sigma h^{3k+1/2} + h^{-5/2}\|\xi_{v_\sigma^\pm}\|^3_{0,\Omega} + C_\sigma h^{2k+2} + \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} \\ &\le C_\sigma h^{2k+1} + \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} + h^{-5/2}\|\xi_{v_\sigma^\pm}\|^3_{0,\Omega}. \end{aligned} \tag{4.25}$$

Finally, for the global Lax–Friedrich flux (3.11), we obtain

$$\begin{aligned} T_{222} &\le \sum_{K\in\mathcal{M}_h}\sum_{e\in\partial K}\left\|\{\!\{\eta_{v_\sigma^\pm}\}\!\}\right\|_{0,e}\|\xi_{v_\sigma^\pm}\|_{0,e}\left\{2\|v_\sigma^\pm\|_{0,\infty,\Omega} \;+\; \|v_\sigma^\pm - v_{\sigma h}^\pm\|_{0,\infty,\Omega}\right\} \\ &= \widetilde{T}_{2221} + \widetilde{T}_{2222} \\ &\le Ch^{-1}\|v_\sigma^\pm\|_{0,\infty,\Omega}\|\eta_{v_\sigma^\pm}\|_{0,\Omega}\|\xi_{v_\sigma^\pm}\|_{0,\Omega} + \widetilde{T}_{2222} \\ &\le C_\sigma h^{2k} + \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} + \widetilde{T}_{2222}, \end{aligned} \tag{4.26}$$

where the term $\widetilde{T}_{2222}$ is bounded as T_{2222}. We then look at the terms T_{23} and T_{24}. Using inverse inequalities and high-order approximation properties for the projection-interpolation operators (Section 3.1.3), the

Cauchy–Schwarz estimate and the Young inequality, we obtain for the term T_{23},

$$
\begin{aligned}
T_{23} &\le \|\xi_{v_\sigma^\pm}\|_{0,\infty,\Omega}\|\eta_{v_\sigma^\pm}\|^2_{0,\mathcal{E}_\parallel} \\
&\le C_\sigma h^{2k-1/2}\|\xi_{v_\sigma^\pm}\|_{0,\Omega} \\
&\le C_\sigma h^{3k+1/2} + h^{-5/2}\|\xi_{v_\sigma^\pm}\|^3_{0,\Omega} \\
&\le C_\sigma h^{2k+1} + h^{-5/2}\|\xi_{v_\sigma^\pm}\|^3_{0,\Omega},
\end{aligned} \tag{4.27}
$$

whereas the term T_{24} is bounded as

$$
\begin{aligned}
T_{24} &\le Ch^{-1}\|\eta_{v_\sigma^\pm}\xi_{v_\sigma^\pm}\|_{0,\Omega}\|\xi_{v_\sigma^\pm}\|_{0,\Omega} \\
&\le C_\sigma h^{k-3/2}\|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} \\
&\le C_\sigma h^{3k+1/2} + h^{-5/2}\|\xi_{v_\sigma^\pm}\|^3_{0,\Omega} \\
&\le C_\sigma h^{2k+1} + h^{-5/2}\|\xi_{v_\sigma^\pm}\|^3_{0,\Omega}.
\end{aligned} \tag{4.28}
$$

Finally, for the term T_{25}, using the Cauchy–Schwarz and inverse inequalities, we obtain

$$
\begin{aligned}
T_{25} &\le \sum_{K\in\mathcal{M}_h}\sum_{e\in\partial K}\left\| [\![\xi_{v_\sigma^\pm}]\!]_\parallel \right\|^2_{0,4,e}\|\xi_{v_\sigma^\pm}\|_{0,e} \\
&\le Ch^{-1}\|\xi_{v_\sigma^\pm}\|^2_{0,4,\Omega}\|\xi_{v_\sigma^\pm}\|_{0,\Omega} \\
&\le Ch^{-5/2}\|\xi_{v_\sigma^\pm}\|^3_{0,\Omega}.
\end{aligned} \tag{4.29}
$$

Gathering estimates of the terms T_1, T_{121}, T_{13}, T_{14}, T_{21}, T_{222}, T_{2221}, T_{2222}, T_{23}, T_{24} and T_{25}, which are given by (4.18)–(4.29), multiplying them by $\mathcal{A}_\sigma$ and summing σ (respectively β) over the bag (respectively branch) set $\mathfrak{L}^{\mathrm{NM}}$ (respectively $\{-,+\}$), we obtain the estimate of Lemma 4.11. □

REMARK 4.12 Let us note that we have also the estimates

$$
T_{14},\ T_{2222},\ T_{24} \lesssim h^{k-3/2}\|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} \lesssim \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} \quad \text{if } k \ge 3/2,
$$

and

$$
T_{13},\ T_{21},\ T_{23} \lesssim h^{2k-1/2}\|\xi_{v_\sigma^\pm}\|_{0,\Omega} \lesssim h^{4k-1} + \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} \lesssim \begin{cases} h^{2k+1} + \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} & \text{if } k \ge 1, \\ h^{2k+2} + \|\xi_{v_\sigma^\pm}\|^2_{0,\Omega} & \text{if } k \ge 3/2. \end{cases}
$$

These estimates could be very useful to show convergence with optimal order $k+1$ or even $k+1/2$. Nevertheless, we are restricted by the estimate of the worse term T_{25}, which behaves like $h^{-5/2}\|\xi_{v_\sigma^\pm}\|^3_{0,\Omega}$. Actually, it seems difficult to improve the estimate of the term T_{25}, by reducing, for example, the power 5/2 to 3/2 or even 2 because it is linked to the dimension (here 3) of the problem. Let us note that if we could obtain an error estimate of order $\min(k,k')+1$ then the presence of the term T_{25} would reduce the condition $k \ge 5/2$ and $k' \ge 5/2$ to $k \ge 3/2$ and $k' \ge 3/2$ (see Section 4.3.4).

4.3.4 *End of the proof* Eventually, from Lemmas 4.5, 4.6, 4.9 and 4.11 we find that the final estimate is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\beta,\sigma}\mathcal{A}_\sigma\|\xi_{v_\sigma^\beta}\|_{0,\Omega}^2\right)$$
$$+\kappa_\perp\sum_{\beta,\sigma}\mathcal{A}_\sigma\int_{\mathscr{E}_{h_\perp}}\mathcal{D}_{\perp,e}\left|\left[\!\left[\xi_{v_\sigma^\beta}\right]\!\right]_\perp\cdot\nu_e\right|^2\mathrm{d}\gamma+\kappa_\parallel\sum_{\beta,\sigma}\mathcal{A}_\sigma\int_{\mathcal{E}_{h_\parallel}}\mathcal{D}^\beta_{\parallel\sigma,e}\left|\left[\!\left[\xi_{v_\sigma^\beta}\right]\!\right]_\parallel\cdot\nu_e\right|^2\mathrm{d}\gamma$$
$$\le C_\odot h^{2\min(k',k)}+C_\odot\sum_{\beta,\sigma}\mathcal{A}_\sigma\left(\|\xi_{v_\sigma^\beta}\|_{0,\Omega}^2+h^{-5/2}\|\xi_{v_\sigma^\beta}\|_{0,\Omega}^3\right).$$

The constant $C_\odot$ has the same dependence as the constant $C_\star$ of Theorem 4.1. If we suppose that $v_{0\sigma h}^\pm=\Pi_h v_{0\sigma}^\pm$ and set

$$X(t)=2C_\odot Th^{2\min(k',k)}+2C_\odot\int_0^t\sum_{\beta,\sigma}\mathcal{A}_\sigma\left(\|\xi_{v_\sigma^\beta}(\tau)\|_{0,\Omega}^2+h^{-5/2}\|\xi_{v_\sigma^\beta}(\tau)\|_{0,\Omega}^3\right)\mathrm{d}\tau, \tag{4.30}$$

then we obtain, for all $\tau\in[0,t]$,

$$\sum_{\beta,\sigma}\mathcal{A}_\sigma\|\xi_{v_\sigma^\beta}(\tau)\|_{0,\Omega}^2\le X(\tau). \tag{4.31}$$

Differentiating (4.30) with respect to time, by (4.31) and using the Cauchy–Schwarz inequality, we infer that

$$\sum_{\beta,\sigma}\mathcal{A}_\sigma\|\xi_{v_\sigma^\beta}(\tau)\|_{0,\Omega}^3\le\left(\sum_{\beta,\sigma}\mathcal{A}_\sigma^{1/3}\|\xi_{v_\sigma^\beta}\|_{0,\Omega}\right)^3\le\left(\sum_{\beta,\sigma}\mathcal{A}_\sigma^{-1/3}\right)^{3/2}\left(\sum_{\beta,\sigma}\mathcal{A}_\sigma\|\xi_{v_\sigma^\beta}\|_{0,\Omega}^2\right)^{3/2}$$

and thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t)=2C_\odot\sum_{\beta,\sigma}\mathcal{A}_\sigma\left(\|\xi_{v_\sigma^\beta}\|_{0,\Omega}^2+h^{-5/2}\|\xi_{v_\sigma^\beta}\|_{0,\Omega}^3\right)\le C_\odot^\diamond\left(X(t)+h^{-5/2}X^{3/2}(t)\right), \tag{4.32}$$

where $C_\odot^\diamond:=2C_\odot\max(1,C^\diamond)$, with $C^{\diamond\,2/3}=\sum_{\beta,\sigma}\mathcal{A}_\sigma^{-1/3}$. Setting the change of variable $Y(t)=X(t)/X(0)$, with $X(0)=2C_\odot Th^{2\min(k',k)}$ and the notation $Y_T=Y(T)$, and introducing the function $y\mapsto f(y):=\int_1^y 1/(s+c(h)s^{3/2})\,\mathrm{d}s$ with $c(h)=X(0)^{1/2}h^{-5/2}=\sqrt{2C_\odot T}h^{\min(k',k)-5/2}$, by (4.32), we obtain

$$f(Y(T))=\int_1^{Y_T}\frac{\mathrm{d}y}{y+c(h)y^{3/2}}=\int_0^{\mathrm{T}}\frac{X'(t)\,\mathrm{d}t}{X(t)+h^{-5/2}X^{3/2}(t)}\le C_\odot^\diamond T.$$

Since $0\le f'(y)\le 1$ for all $y\in[1,\infty[$ uniformly with respect to h as long as $k\ge 5/2$ and $k'\ge 5/2$, then the function f is monotonically nondecreasing and positive ($f(1)=0$) on the interval $y\in[1,\infty[$ and thus invertible on the same interval. Since $f(Y(t))\le f(Y(T))\le C_\odot^\diamond T$, we get $Y(t)\le f^{-1}(C_\odot^\diamond T)$, i.e.,

$X(t) \le X(0) f^{-1}(C_{\odot}^{\lozenge} T) \le f^{-1}(C_{\odot}^{\lozenge} T)\sqrt{2C_{\odot}T}h^{2\min(k,k')}$. Therefore, there exists a constant $C_\star$ as described in Theorem 4.1 such that

$$\|\xi_{v_\sigma^\pm}\|_{L^\infty\left(0,T;L^2(\Omega)\right)} \le C_\star h^{\min(k,k')} \quad \forall \sigma \in \mathfrak{L}^{\mathrm{NM}}.$$

Using this last estimate and the high-order approximation property of the L^2-projection operator Π_h (Section 3.1.3), we obtain the first estimate of Theorem 4.1 for the approximation of contours. The second estimate of Theorem 4.1, for the electrical potential approximation, is obtained from the estimate of Proposition 4.3 with $|\alpha| = 0$ and the first estimate of Theorem 4.1.

4.4 *Conservation and stability properties of the numerical scheme*

In this section, we give some stability properties of the contour approximations in L^2 (see Lemma 4.14) and L^∞ (see Corollary 4.15) norms but also for the electrical potential approximation in $L^2\left(\Omega_\parallel;\, H^1(\Omega_\perp)\right)$ and $L^\infty\left(\Omega_\parallel;\, W^{1,\infty}(\Omega_\perp)\right)$ norms (see Corollary 4.16). We start with a lemma giving charge or mass conservation.

LEMMA 4.13 (Mass conservation). The numerical scheme constituted by the equations (3.9), (3.13) and (3.14) preserves the mass (or the number of particles). Actually, for all $t \in [0, T]$, the approximate solution $v_{\sigma h}^\pm$ satisfies

$$\int_\Omega v_{\sigma h}^\pm(t)\,\mathrm{d}x = \int_\Omega v_{\sigma h}^\pm(0)\,\mathrm{d}x \quad \forall \sigma \in \mathfrak{L}^{\mathrm{NM}},$$

hence mass is preserved, i.e.,

$$\int_\Omega \mathrm{d}x \sum_{\sigma\in\mathfrak{L}^{\mathrm{NM}}} \mathcal{A}_\sigma (v_{\sigma h}^+(t) - v_{\sigma h}^-(t)) = \int_\Omega \mathrm{d}x \sum_{\sigma\in\mathfrak{L}^{\mathrm{NM}}} \mathcal{A}_\sigma (v_{\sigma h}^+(0) - v_{\sigma h}^-(0)).$$

Proof. Taking $\varphi_h = 1$ (on each cell K and 0 elsewhere) in (3.9), the compensation of boundary terms on each cell and cancellation of gradient terms lead to the desired result after summation over all cells of the mesh. □

We continue with the L^2-stability of the approximation of contours given by the following lemma.

LEMMA 4.14 (L^2-stability). The approximate solution $v_{\sigma h}^\pm$ given by the scheme (3.9) and (3.13)–(3.14) satisfies the L^2-bound

$$\|v_{\sigma h}^\pm(t)\|_{0,\Omega} \le C_v \quad \forall t \in [0, T], \quad \forall \sigma \in \mathfrak{L}^{\mathrm{NM}},$$

where the constant C_v depends on the final time T, the norm $\|v_{\sigma h}^\pm(0)\|_{0,\Omega}$ and the constant C_2 appearing in Proposition 4.3.

Proof. Substituting $\varphi_h = v^{\pm}_{\sigma h}$ in (3.9), after an integration by parts and using estimate $|\left[\!\left[v^{\pm}_{\sigma}\right]\!\right]_{\parallel}| \leq 2\mathcal{D}^{\pm}_{\parallel\sigma,e}$ we get

$$\left\langle\!\left\langle {v^{\pm}_{\sigma h}}^2/2 \right\rangle\!\right\rangle \left[\!\left[v^{\pm}_{\sigma h}\right]\!\right]_{\parallel} - \frac{1}{6}\left[\!\left[{v^{\pm}_{\sigma h}}^3\right]\!\right]_{\parallel} = \frac{1}{2}\left|\left[\!\left[v^{\pm}_{\sigma h}\right]\!\right]_{\parallel}\right|^2 \left(\mathcal{D}^{\pm}_{\parallel\sigma,e} + \frac{1}{6}\left[\!\left[v^{\pm}_{\sigma h}\right]\!\right]_{\parallel}\right) \geq \frac{1}{3}\left|\left[\!\left[v^{\pm}_{\sigma h}\right]\!\right]_{\parallel}\cdot \nu_e\right|^2 \mathcal{D}^{\pm}_{\parallel\sigma,e}.$$

Therefore, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^{\pm}_{\sigma h}\|_{0,\Omega} &+ \frac{1}{2}\int_{\mathscr{E}_{h_\perp}} \mathcal{D}_{\perp,e}\left|\left[\!\left[v^{\pm}_{\sigma h}\right]\!\right]_{\perp}\cdot \nu_e\right|^2 \mathrm{d}\gamma + \frac{1}{3}\int_{\mathcal{E}_{h_\parallel}} \mathcal{D}^{\pm}_{\parallel\sigma,e}\left|\left[\!\left[v^{\pm}_{\sigma h}\right]\!\right]_{\parallel}\cdot \nu_e\right|^2 \mathrm{d}\gamma \\ &+ \int_{\mathcal{E}_{h_\parallel}} \left[\!\left[\mathcal{R}_h\mathcal{J}_\perp\phi_h\right]\!\right]_{\parallel}\left\{\{\!\{v^{\pm}_{\sigma h}\}\!\}_\alpha - 2\{\!\{v^{\pm}_{\sigma h}\}\!\}\right\}\mathrm{d}\gamma + \int_\Omega v^{\pm}_{\sigma h}\partial_\parallel \mathcal{R}_h\mathcal{J}_\perp\phi_h\,\mathrm{d}x \leq 0. \end{aligned} \tag{4.33}$$

Now using Proposition 4.3, inverse inequalities, high-order approximation properties for the projection-interpolation operators (Section 3.1.3), stability properties of the gyroaverage operator (Section 3.1.4), the Cauchy–Schwarz estimate and the Young inequality, we obtain

$$\begin{aligned}&\int_{\mathcal{E}_{h_\parallel}} \left[\!\left[\mathcal{R}_h\mathcal{J}_\perp\phi_h\right]\!\right]_{\parallel}\left\{\{\!\{v^{\pm}_{\sigma h}\}\!\}_\alpha - 2\{\!\{v^{\pm}_{\sigma h}\}\!\}\right\}\mathrm{d}\gamma + \int_\Omega v^{\pm}_{\sigma h}\partial_\parallel \mathcal{R}_h\mathcal{J}_\perp\phi_h\,\mathrm{d}x \\
&= \int_{\mathcal{E}_{h_\parallel}} \left[\!\left[\mathcal{R}_h\mathcal{J}_\perp\phi_h - \mathcal{J}_\perp\phi\right]\!\right]_{\parallel}\left\{\{\!\{v^{\pm}_{\sigma h}\}\!\}_\alpha - 2\{\!\{v^{\pm}_{\sigma h}\}\!\}\right\}\mathrm{d}\gamma \\
&\quad + \int_\Omega v^{\pm}_{\sigma h}\partial_\parallel(\mathcal{R}_h\mathcal{J}_\perp\phi_h - \mathcal{J}_\perp\phi)\,\mathrm{d}x + \int_\Omega v^{\pm}_{\sigma h}\partial_\parallel\mathcal{J}_\perp\phi\,\mathrm{d}x \\
&\leq \sum_{K\in\mathcal{M}_h}\left\{\sum_{e\in\partial K}\|\mathcal{R}_h\mathcal{J}_\perp\phi_h - \mathcal{J}_\perp\phi\|_{0,e}\|v^{\pm}_{\sigma h}\|_{0,e} + \|v^{\pm}_{\sigma h}\|_{0,K}\left(\|\partial_\parallel(\mathcal{R}_h\mathcal{J}_\perp\phi_h - \mathcal{J}_\perp\phi)\|_{0,K} + \|\partial_\parallel\mathcal{J}_\perp\phi\|_{0,K}\right)\right\} \\
&\leq \sum_{K\in\mathcal{M}_h}\left\{\sum_{e\in\partial K}\left(\|(I-\mathcal{R}_h)\mathcal{J}_\perp\phi\|_{0,e} + \|\mathcal{R}_h\mathcal{J}_\perp(\phi-\phi_h)\|_{0,e}\right)\|v^{\pm}_{\sigma h}\|_{0,e}\right. \\
&\quad \left. + \|v^{\pm}_{\sigma h}\|_{0,K}\left(\|\partial_\parallel(I-\mathcal{R}_h)\mathcal{J}_\perp\phi\|_{0,K} + \|\partial_\parallel\mathcal{R}_h\mathcal{J}_\perp(\phi-\phi_h)\|_{0,K} + \|\partial_\parallel\mathcal{J}_\perp\phi\|_{0,K}\right)\right\} \\
&\leq \|v^{\pm}_{\sigma h}\|_{0,\Omega}\left(Ch^{k'}\|\phi\|_{k'+1,\Omega} + \|\phi\|_{1,\Omega} + h^{-1}\|\phi-\phi_h\|_{0,\Omega}\right) \\
&\leq \|v^{\pm}_{\sigma h}\|_{0,\Omega}\left(Ch^{k'}\|\phi\|_{k'+1,\Omega} + \|\phi\|_{1,\Omega} + C_2h^{\min(k,k')} + \left(\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v^\beta_\sigma - v^\beta_{\sigma h}\right\|^2_{0,\Omega}\right)^{1/2}\right) \\
&\leq C\|v^{\pm}_{\sigma h}\|_{0,\Omega}\left(1 + \left(\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v^\beta_{\sigma h}\right\|^2_{0,\Omega}\right)^{1/2}\right) \\
&\leq \|v^{\pm}_{\sigma h}\|^2_{0,\Omega} + C\left(1 + \left(\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v^\beta_{\sigma h}\right\|^2_{0,\Omega}\right)\right).
\end{aligned}$$

Substituting the last inequality in (4.33), multiplying the result by $\mathcal{A}_\sigma$ and summing σ (respectively β) over the bag (respectively branch) set $\mathfrak{L}^{\mathrm{NM}}$ (respectively $\{-,+\}$), we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v^\beta_{\sigma h}\right\|^2_{0,\Omega}\right) \le \sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v^\beta_{\sigma h}\right\|^2_{0,\Omega} + C\left(1+\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v^\beta_{\sigma h}\right\|^2_{0,\Omega}\right).$$

Setting

$$X(t) = 1 + \sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v^\beta_{\sigma h}(t)\right\|^2_{0,\Omega},$$

the last differential inequality becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) \le CX(t).$$

Using the Gronwall lemma, this differential inequality leads to estimate (4.14) of Lemma 4.14. □

COROLLARY 4.15 (L^∞-stability). Let $v^\pm_{\sigma h}$ be the approximate solution given by the scheme (3.9) and (3.13)–(3.14). Then there exist positive constants m^*_μ and M^*_μ, independent of h, such that for all time $t \in [0,T]$, for every $\mu \in \mathfrak{M}^{\mathrm{M}}$, the map $a \mapsto v^-_{a\mu h}$ (respectively $a \mapsto v^+_{a\mu h}$) is nondecreasing (respectively nonincreasing) on the set $\mathcal{A}^{\mathrm{N}}$, such that

$$m^*_\mu < \pm v^\pm_{a\mu h}(t) < M^*_\mu, \quad \forall t \in [0,T] \quad \forall a \in \mathcal{A}^{\mathrm{N}}.$$

Proof. Using inverse inequalities and high-order approximation properties for the projection-interpolation operators (Section 3.1.3), from Theorem 4.1, we get

$$\begin{aligned}\|v^\pm_\sigma - v^\pm_{\sigma h}\|_{0,\infty,\Omega} &\le \|\Pi_h v^\pm_\sigma - v^\pm_{\sigma h}\|_{0,\infty,\Omega} + \|v^\pm_\sigma - \Pi_h v^\pm_\sigma\|_{0,\infty,\Omega}\\ &\le h^{-3/2}\|\Pi_h v^\pm_\sigma - v^\pm_{\sigma h}\|_{0,\Omega} + Ch|v^\pm_\sigma|_{1,\infty,\Omega}\\ &\le C_\star h^{\min(1,k-3/2,k'-3/2)} \le C_\star h^\eta,\end{aligned}$$

with $0 < \eta < 1$ as long as $k,\ k' > 3/2$. Therefore, we get $v^\pm_{\sigma h} = v^\pm_\sigma + \mathcal{O}(h^\eta)$, and for h small enough,

$$0 < m^*_\mu \le m_\mu - C_\star h^\eta < \pm v^\pm_{\sigma h} < M_\mu + C_\star h^\eta \le M^*_\mu < \infty,$$

which ends the proof. □

COROLLARY 4.16 The approximate solution ϕ_h given by the scheme (3.9) and (3.13)–(3.14) satisfies the bounds

$$\|\phi_h(t)\|_{0,\Omega},\ \|\phi_h(t)\|_{0,\infty,\Omega},\ \|\phi_h(t)\|_{L^2(\Omega_\parallel;H^1(\Omega_\perp))},\ \|\phi_h(t)\|_{L^\infty(\Omega_\parallel;W^{1,\infty}(\Omega_\perp))} \le C_\phi \quad \forall t \in [0,T],$$

where the constant C_ϕ depends on the constant $C_\star$ appearing in Theorem 4.1.

Proof. The estimates of Corollary 4.16 follow straightforwardly from Propositions 4.3 and 4.4, and the error estimate for the approximation of contours stated in Theorem 4.1. □

5. The total energy is asymptotically nonincreasing

In this section, we prove that the total energy of the approximate solution is asymptotically nonincreasing.

THEOREM 5.1 Assume $k \geq 5/2$, $k' \geq 5/2$ and $k' \leq k$. Let $(v^{\pm}_{\sigma h}, \phi_h) \in V^k_h \times \Phi^{k'}_h \cap L^2\left(\Omega_\parallel; H^1_0(\Omega_\perp)\right)$ be the approximate solution of (2.8)–(2.9) given by the DGFEM scheme (3.9) and (3.13)–(3.14), described in Section 3.2. Then the total energy of the approximate solution is asymptotically nonincreasing in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{6}\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v^\beta_{\sigma h}\right\|^3_{0,\Omega}+\frac{1}{2}\int_\Omega\left(a_0|\nabla_\perp\phi_h|^2+b_0|\phi_h-\bar{\phi}_h|^2\right)\mathrm{d}x\right)\leq C_{\mathcal{J}_\perp}h^{\min(k,k')}. \tag{5.1}$$

Here, the constant $C_{\mathcal{J}_\perp}=0$ when $\mathcal{J}_\perp=I$ (drift-kinetic case) and $C_{\mathcal{J}_\perp}>0$ otherwise (gyrokinetic case). The constant $C_{\mathcal{J}_\perp}$ depends on the constants M^*_μ, C_v, C_ϕ and $C_\star$. For $\phi_h \in \widehat{\Phi}^{k'}_h$, estimate (5.1) holds for any flux defined in Section 3.2.1, while for $\phi_h \in \widehat{\Phi}^{k'}_h$, estimate (5.1) holds only with $\alpha = 1/2$ in the flux (3.12) and $\mathcal{D}^\beta_{\parallel\sigma,e}=\mathcal{D}_{\parallel,e}=$ constant >0 (i.e., uniform with respect to σ, β and $x_\perp$) in the flux (3.11).

REMARK 5.2 The present schemes are not energy preserving. Recently, some energy-conserving DG schemes have been developed for *canonical* Vlasov equations such as Vlasov–Poisson, Vlasov–Ampère and Vlasov–Maxwell (Ayuso de Dios *et al.*, 2011, 2012; Cheng *et al.*, 2014b,c; Madaule *et al.*, 2014). Unfortunately these schemes seem not to work directly for the gyrokinetic-Vlasov equations, because they rely on the canonical structure of Vlasov–Poisson, Vlasov–Ampère and Vlasov–Maxwell. Indeed the property, according to which the force field is componentwise divergence-free, plays an important part in these energy-conserving schemes. This canonical structure and the componentwise divergence-free property of the force-field are lost for the gyrokinetic-Vlasov system, because the latter is usually written in noncanonical variables. For the gyrokinetic-waterbag equations, it is even worse since the structure of the transport equation is closer to nonlinear hyperbolic conservation laws of fluid mechanics than collisionless kinetic equations (Vlasov). Nevertheless, we can think about two ideas for retrieving energy-preserving schemes. The first one is to obtain more degrees of freedom for the choice of numerical fluxes by adopting fully discontinuous Galerkin schemes (in parallel and perpendicular directions) for the electrical potential and drift velocity. These new degrees of freedom could be used to obtain a modified consistent electrical potential, which will enforce a modified energy conservation law such as in Ayuso de Dios *et al.* (2011, 2012); Cheng *et al.* (2014b,c); Madaule *et al.* (2014). Let us note that the modified energy conservation laws found in Ayuso de Dios *et al.* (2011, 2012); Cheng *et al.* (2014b,c); Madaule *et al.* (2014) differ from the unmodified original ones by small terms of order h^p (with p the order of the scheme), just as in Theorem 5.1. The second one is to couple the gyrokinetic-waterbag equations with the following energy conservation equation to obtain modified contours and/or electrical potential that satisfy the energy conservation law (5.2):

$$\partial_t\left(\frac{e_\sigma}{2}\right)+\nabla_\perp\cdot\left(\frac{e_\sigma}{2}\mathcal{J}_\perp v_E\right)+\partial_\parallel\left(u_\sigma\left[\frac{e_\sigma}{2}+\mathfrak{p}_\sigma\right]\right)+\frac{q_i}{m_i}c_\sigma u_\sigma\partial_\parallel\mathcal{J}_\perp\phi=0.$$

Here, we denote the density by $c_\sigma=\mathcal{A}_\sigma(v^+_\sigma-v^-_\sigma)$, the mean velocity by $u_\sigma=(v^+_\sigma+v^-_\sigma)/2$, the kinetic energy by $e=({v^+_\sigma}^3-{v^-_\sigma}^3)/3$ and the pressure by $\mathfrak{p}_\sigma=c^3_\sigma/(12\mathcal{A}^2_\sigma)$.

Proof. On the one hand, setting $\varphi_h = \mathcal{R}_h\mathcal{J}_\perp\phi_h \in \Phi_h^{k'} \subset V_h^k$ (with $k' \leq k$) in (3.9), summing over all the cells K of the mesh, using the continuity property of $v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nu_{K_\perp}$ across the perpendicular direction to the edges of the cell K for any $K \in \mathcal{M}_h$, we obtain

$$\int_\Omega \partial_t v_{\sigma h}^{\pm} \mathcal{R}_h\mathcal{J}_\perp\phi_h \,\mathrm{d}x - \sum_{K\in\mathcal{M}_h}\int_K \left(\mathcal{R}_h\mathcal{J}_\perp\phi_h + {v_{\sigma h}^{\pm}}^2/2\right)\partial_\parallel \mathcal{R}_h\mathcal{J}_\perp\phi_h\,\mathrm{d}x$$
$$+ \sum_{K\in\mathcal{M}_h}\int_{\partial K} \langle\!\langle \mathcal{R}_h\mathcal{J}_\perp\phi_h\rangle\!\rangle\, (\mathcal{R}_h\mathcal{J}_\perp\phi_h)^\ell \nu_{K_\parallel}\,\mathrm{d}\gamma + \sum_{K\in\mathcal{M}_h}\int_{\partial K}\left\langle\!\!\left\langle {v_{\sigma h}^{\pm}}^2/2\right\rangle\!\!\right\rangle (\mathcal{R}_h\mathcal{J}_\perp\phi_h)^\ell\nu_{K_\parallel}\,\mathrm{d}\gamma = 0. \tag{5.2}$$

On the other hand, setting $\varphi_h = {v_{\sigma h}^{\pm}}^2/2$, in (3.9), after an integration by parts and a sum over all the cells K of the mesh, we obtain

$$\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega {v_{\sigma h}^{\pm}}^3\,\mathrm{d}x + \int_{\mathscr{E}_{h_\perp}} \left\{\left\langle\!\!\left\langle v_{\sigma h}^{\pm} v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\right\rangle\!\!\right\rangle\cdot \left[\!\left[{v_{\sigma h}^{\pm}}^2/2\right]\!\right]_\perp - v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\left[\!\left[{v_{\sigma h}^{\pm}}^3/3\right]\!\right]_\perp\right\}\mathrm{d}\gamma$$
$$+\int_{\mathcal{E}_{h_\parallel}}\left\{\frac{1}{2}\left\langle\!\!\left\langle {v_{\sigma h}^{\pm}}^2/2\right\rangle\!\!\right\rangle \left[\!\left[{v_{\sigma h}^{\pm}}^2\right]\!\right]_\parallel - \frac{1}{8}\left[\!\left[{v_{\sigma h}^{\pm}}^4\right]\!\right]_\parallel\right\}\mathrm{d}\gamma$$
$$+\sum_{K\in\mathcal{M}_h}\left\{\int_{\partial K}\langle\!\langle \mathcal{R}_h\mathcal{J}_\perp\phi_h\rangle\!\rangle\,({v_{\sigma h}^{\pm}}^2/2)^\ell\nu_{K_\parallel}\,\mathrm{d}\gamma - \int_{\partial K}\left(\frac{1}{2}{v_{\sigma h}^{\pm}}^2\mathcal{R}_h\mathcal{J}_\perp\phi_h\right)^\ell \nu_{K_\parallel}\,\mathrm{d}\gamma\right\}$$
$$+\sum_{K\in\mathcal{M}_h}\int_K \frac{1}{2}{v_{\sigma h}^{\pm}}^2\partial_\parallel\mathcal{R}_h\mathcal{J}_\perp\phi_h\,\mathrm{d}x = 0. \tag{5.3}$$

Substituting (5.2) to replace the last term of (5.3), using the flux definitions of Section 3.2.1, taking into account that $\pm v_{\sigma h}^{\pm} > 0$ from Corollary 4.15, multiplying equation (5.3) by $\beta\mathcal{A}_\sigma$ and summing σ (respectively β) over the bag (respectively branch) set $\mathfrak{L}^{\mathrm{NM}}$ (respectively $\{-,+\}$), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{6}\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v_{\sigma h}^\beta\right\|_{0,\Omega}^3\right) + \sum_\sigma \mathcal{A}_\sigma\int_\Omega \partial_t(v_{\sigma h}^+ - v_{\sigma h}^-)\mathcal{R}_h\mathcal{J}_\perp\phi_h\,\mathrm{d}x$$
$$+\sum_{\beta,\sigma}\mathcal{A}_\sigma\int_{\mathcal{E}_h}\left\{\frac{\beta}{2}\left[\!\left[v_{\sigma h}^\beta\right]\!\right]^2\left(\{\!\{v_{\sigma h}^\beta\}\!\}(\mathcal{D}_{\perp,e}+\mathcal{D}_{\parallel\sigma,e}^\beta) - \frac{1}{6}v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\left[\!\left[v_{\sigma h}^\beta\right]\!\right]_\perp\right)\right\}\mathrm{d}\gamma$$
$$+\sum_{\beta,\sigma}\mathcal{A}_\sigma\int_{\mathcal{E}_{h_\parallel}}\beta\left[\!\left[\mathcal{R}_h\mathcal{J}_\perp\phi_h\right]\!\right]_\parallel\left[\!\left[v_{\sigma h}^\beta\right]\!\right]_\parallel\left(\frac{1}{2}\mathcal{D}_{\parallel\sigma,e}^\beta + (\alpha - 1/2)\{\!\{v_{\sigma h}^\beta\}\!\}\right)\mathrm{d}\gamma = 0, \tag{5.4}$$

where $\mathcal{D}_{\perp,e} = 0$ if $e \in \mathcal{E}_{h_\parallel}$ and $\mathcal{D}_{\parallel\sigma,e}^\beta = 0$ if $e \in \mathscr{E}_{h_\perp}$. Now, let us rewrite the second term of (5.4). Differentiating (3.14) with respect to time and next setting $\psi_h = \phi_h$, we obtain

$$\int_\Omega \phi_h\Pi_h\mathcal{J}_\perp\partial_t\rho_h\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(a_0|\nabla_\perp\phi_h|^2 + b_0|\phi_h - \bar{\phi}_h|^2\right)\mathrm{d}x. \tag{5.5}$$

Using the properties of the L^2-projection operator Π_h, $\phi_h \in \Phi_h^{k'} \subset V_h^k$ (with $k' \leq k$), $\partial_t \rho_h \in V_h^k$, equation (5.5) and the property $\int_{\Omega_\perp} \varphi \mathcal{J}_\perp \psi \,\mathrm{d}x_\perp = \int_{\Omega_\perp} \psi \mathcal{J}_\perp \varphi \,\mathrm{d}x_\perp$, we obtain

$$\begin{aligned}
\int_\Omega \partial_t \rho_h \mathcal{R}_h \mathcal{J}_\perp \phi_h \,\mathrm{d}x &= \int_\Omega \partial_t \rho_h \Pi_h \mathcal{J}_\perp \phi_h \,\mathrm{d}x - \int_\Omega \partial_t \rho_h (\Pi_h - \mathcal{R}_h) \mathcal{J}_\perp \phi_h \,\mathrm{d}x \\
&= \int_\Omega \partial_t \rho_h \mathcal{J}_\perp \phi_h \,\mathrm{d}x - \int_\Omega \partial_t \rho_h (\Pi_h - \mathcal{R}_h) \mathcal{J}_\perp \phi_h \,\mathrm{d}x \\
&= \int_\Omega \phi_h \mathcal{J}_\perp \partial_t \rho_h \,\mathrm{d}x - \int_\Omega \partial_t \rho_h (\Pi_h - \mathcal{R}_h) \mathcal{J}_\perp \phi_h \,\mathrm{d}x \\
&= \int_\Omega \phi_h \Pi_h \mathcal{J}_\perp \partial_t \rho_h \,\mathrm{d}x - \int_\Omega \partial_t \rho_h (\Pi_h - \mathcal{R}_h) \mathcal{J}_\perp \phi_h \,\mathrm{d}x \\
&= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(a_0 |\nabla_\perp \phi_h|^2 + b_0 |\phi_h - \bar{\phi}_h|^2 \right) \mathrm{d}x - \int_\Omega \partial_t \rho_h (\Pi_h - \mathcal{R}_h) \mathcal{J}_\perp \phi_h \,\mathrm{d}x.
\end{aligned}$$

From the last equality, (5.4) becomes

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{6} \sum_{\beta,\sigma} \mathcal{A}_\sigma \left\| v_{\sigma h}^\beta \right\|_{0,\Omega}^3 + \frac{1}{2} \int_\Omega \left(a_0 |\nabla_\perp \phi_h|^2 + b_0 |\phi_h - \bar{\phi}_h|^2 \right) \mathrm{d}x \right) \\
&\quad + \sum_{\beta,\sigma} \mathcal{A}_\sigma \int_{\mathcal{E}_h} \left\{ \frac{\beta}{2} \left[\!\left[v_{\sigma h}^\beta \right]\!\right]^2 \left(\{\!\{ v_{\sigma h}^\beta \}\!\} (\mathcal{D}_{\perp,e} + \mathcal{D}_{\parallel\sigma,e}^\beta) - \frac{1}{6} v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \cdot \left[\!\left[v_{\sigma h}^\beta \right]\!\right]_\perp \right) \right\} \mathrm{d}\gamma \\
&\quad + \sum_{\beta,\sigma} \mathcal{A}_\sigma \int_{\mathcal{E}_{h_\parallel}} \beta \left[\!\left[\mathcal{R}_h \mathcal{J}_\perp \phi_h \right]\!\right]_\parallel \left[\!\left[v_{\sigma h}^\beta \right]\!\right]_\parallel \left(\frac{1}{2} \mathcal{D}_{\parallel\sigma,e}^\beta + (\alpha - 1/2) \{\!\{ v_{\sigma h}^\beta \}\!\} \right) \mathrm{d}\gamma \\
&= \int_\Omega \partial_t \rho_h (\Pi_h - \mathcal{R}_h) \mathcal{J}_\perp \phi_h \,\mathrm{d}x.
\end{aligned} \tag{5.6}$$

We then estimate the right-hand side of (5.6). For this purpose, we set $\varphi_h = (\Pi_h - \mathcal{R}_h)\mathcal{J}_\perp \phi_h \in V_h^k$ (with $k' \leq k$) in (3.9), we multiply the result by $\beta \mathcal{A}_\sigma$ and sum σ (respectively β) over the bag (respectively branch) set $\mathfrak{L}^{\mathrm{NM}}$ (respectively $\{-,+\}$) to obtain

$$\begin{aligned}
&\int_\Omega \partial_t \rho_h (\Pi_h - \mathcal{R}_h) \mathcal{J}_\perp \phi_h \,\mathrm{d}x - \sum_{K \in \mathcal{M}_h} \int_K \rho_h v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \cdot \nabla_\perp (\Pi_h - \mathcal{R}_h) \mathcal{J}_\perp \phi_h \,\mathrm{d}x \\
&\quad + \sum_{K \in \mathcal{M}_h} \sum_{\beta,\sigma} \mathcal{A}_\sigma \int_{\partial K} \beta \left\langle\!\left\langle v_{\sigma h}^\beta v_E(\mathcal{R}_h \mathcal{J}_\perp \phi_h) \cdot \nu_{K_\perp} \right\rangle\!\right\rangle ([\Pi_h - \mathcal{R}_h] \mathcal{J}_\perp \phi_h)^\ell \,\mathrm{d}\gamma \\
&\quad - \sum_{K \in \mathcal{M}_h} \sum_{\beta,\sigma} \mathcal{A}_\sigma \int_K \frac{\beta}{2} (v_{\sigma h}^\beta)^2 \partial_\parallel (\Pi_h - \mathcal{R}_h) \mathcal{J}_\perp \phi_h \,\mathrm{d}x \\
&\quad + \sum_{K \in \mathcal{M}_h} \sum_{\beta,\sigma} \mathcal{A}_\sigma \int_{\partial K} \beta \left\langle\!\left\langle (v_{\sigma h}^\beta)^2 / 2 \right\rangle\!\right\rangle \nu_{K_\parallel} ([\Pi_h - \mathcal{R}_h] \mathcal{J}_\perp \phi_h)^\ell \,\mathrm{d}\gamma = 0.
\end{aligned} \tag{5.7}$$

Using inverse inequalities, stability and high-order approximation properties for the projection-interpolation operators (Section 3.1.3), stability properties of the gyroaverage operator (Section 3.1.4), Lemma 4.14, Corollaries 4.15 and 4.16, Proposition 4.3 and Theorem 4.1, we obtain for the second term of the left-hand side of (5.7),

$$
\begin{aligned}
&\sum_{K\in\mathcal{M}_h}\int_K \rho_h v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nabla_\perp(\Pi_h-\mathcal{R}_h)\mathcal{J}_\perp\phi_h\,\mathrm{d}x\\
&\quad\le \|\rho_h\|_{0,\infty,\Omega}\sum_{K\in\mathcal{M}_h}\|v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\|_{0,K}\|\nabla_\perp(\Pi_h-\mathcal{R}_h)\mathcal{J}_\perp\phi_h\|_{0,K}\\
&\quad\le \|\rho_h\|_{0,\infty,\Omega}\Big(\sum_{K\in\mathcal{M}_h}\|\mathcal{R}_h\mathcal{J}_\perp\phi_h\|^2_{L^2(K_\parallel;H^1(K_\perp))}\Big)^{1/2}\Big(\sum_{K\in\mathcal{M}_h}\big(\|\nabla_\perp(I-\Pi_h)\mathcal{J}_\perp\phi\|^2_{0,K}\\
&\qquad+\|\nabla_\perp(I-\mathcal{R}_h)\mathcal{J}_\perp\phi\|^2_{0,K}+\|\nabla_\perp(\Pi_h-\mathcal{R}_h)\mathcal{J}_\perp(\phi-\phi_h)\|^2_{0,K}\big)\Big)^{1/2}\\
&\quad\le \|\rho_h\|_{0,\infty,\Omega}\|\phi_h\|_{L^2(\Omega_\parallel;H^1(\Omega_\perp))}\Big(\sum_{K\in\mathcal{M}_h}\big(Ch^{2k}\|\phi\|^2_{k+1,K}+Ch^{2k'}\|\phi\|^2_{k'+1,K}+\|\phi-\phi_h\|^2_{L^2(K_\parallel;H^1(K_\perp))}\big)\Big)^{1/2}\\
&\quad\le C\Big(\|\rho_h\|_{0,\infty,\Omega},\|\phi_h\|_{L^2(\Omega_\parallel;H^1(\Omega_\perp))},\|\phi\|_{\max(k,k')+1,\Omega},C_\star\Big)\,h^{\min(k,k')},
\end{aligned}
\tag{5.8}
$$

and for the third term of the left-hand side of (5.7),

$$
\begin{aligned}
&\sum_{K\in\mathcal{M}_h}\sum_{\beta,\sigma}\mathcal{A}_\sigma\int_{\partial K}\beta\,\Big\langle\!\Big\langle v^\beta_{\sigma h}v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nu_{K_\perp}\Big\rangle\!\Big\rangle\,([\Pi_h-\mathcal{R}_h]\mathcal{J}_\perp\phi_h)^\ell\,\mathrm{d}\gamma\\
&\quad\le\sum_{K\in\mathcal{M}_h}\sum_{\beta,\sigma}\mathcal{A}_\sigma\sum_{e\in\partial K}\|(\Pi_h-\mathcal{R}_h)\mathcal{J}_\perp\phi_h\|_{0,e}\Big\{\|v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\|_{0,\infty,\Omega}\Big\|\big[\!\big[v^\beta_{\sigma h}\big]\!\big]_\perp\cdot\nu_e\Big\|_{0,e}\\
&\qquad+\|v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\nu_e\|_{0,e}\Big\|v^\beta_{\sigma h}\Big\|_{0,\infty,\Omega}\Big\}\\
&\quad\le h^{-1}\Big\{\|(I-\Pi_h)\mathcal{J}_\perp\phi\|_{0,\Omega}+\|(I-\mathcal{R}_h)\mathcal{J}_\perp\phi\|_{0,\Omega}+\|(\Pi_h-\mathcal{R}_h)\mathcal{J}_\perp(\phi-\phi_h)\|_{0,\Omega}\Big\}\Big\{\sqrt{2}\Big(\sum_\sigma\mathcal{A}_\sigma\Big)^{1/2}\\
&\qquad\times\Big(\sum_{\beta,\sigma}\mathcal{A}_\sigma\Big\|v^\beta_{\sigma h}\Big\|^2_{0,\Omega}\Big)^{1/2}\|\phi_h\|_{L^\infty(\Omega_\parallel;W^{1,\infty}(\Omega_\perp))}+\|\phi_h\|_{L^2(\Omega_\parallel;H^1(\Omega_\perp))}\sum_{\beta,\sigma}\mathcal{A}_\sigma\Big\|v^\beta_{\sigma h}\Big\|_{0,\infty,\Omega}\Big\}\\
&\quad\le C\Big(\|\phi_h\|_{L^2(\Omega_\parallel;H^1(\Omega_\perp))},\|\phi_h\|_{L^\infty(\Omega_\parallel;W^{1,\infty}(\Omega_\perp))},\Big\|v^\beta_{\sigma h}\Big\|_{0,\infty,\Omega},\Big\|v^\beta_{\sigma h}\Big\|_{0,\Omega},C_\star\Big)\,h^{\min(k,k')}.
\end{aligned}
\tag{5.9}
$$

The fourth term of the left-hand side of (5.7) is bounded as

$$
\begin{aligned}
-\sum_{K\in\mathcal{M}_h}\sum_{\beta,\sigma}\mathcal{A}_\sigma\int_K \frac{\beta}{2}(v^\beta_{\sigma h})^2\partial_\parallel(\Pi_h-\mathcal{R}_h)\mathcal{J}_\perp\phi_h\,\mathrm{d}x
&\le h^{-1}\|(\Pi_h-\mathcal{R}_h)\mathcal{J}_\perp\phi_h\|_{0,\Omega}\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v^\beta_{\sigma h}\right\|_{0,\infty,\Omega}\left\|v^\beta_{\sigma h}\right\|_{0,\Omega}\\
&\le C\left(\left\|v^\beta_{\sigma h}\right\|_{0,\infty,\Omega},\left\|v^\beta_{\sigma h}\right\|_{0,\Omega},C_\star\right)h^{\min(k,k')}, \qquad (5.10)
\end{aligned}
$$

while the fifth term of the left-hand side of (5.7) is bounded as

$$
\begin{aligned}
&\sum_{K\in\mathcal{M}_h}\sum_{\beta,\sigma}\mathcal{A}_\sigma\int_{\partial K}\beta\left\langle\!\left\langle(v^\beta_{\sigma h})^2/2\right\rangle\!\right\rangle \nu_{K_\parallel}([\Pi_h-\mathcal{R}_h]\mathcal{J}_\perp\phi_h)^\ell\,\mathrm{d}\gamma\\
&\le\sum_{K\in\mathcal{M}_h}\sum_{\beta,\sigma}\mathcal{A}_\sigma\sum_{e\in\partial K}\left\|v^\beta_{\sigma h}\right\|_{0,\infty,\Omega}\left\|v^\beta_{\sigma h}\right\|_{0,e}\|(\Pi_h-\mathcal{R}_h)\mathcal{J}_\perp\phi_h\|_{0,e}\\
&\le h^{-1}\|(\Pi_h-\mathcal{R}_h)\mathcal{J}_\perp\phi_h\|_{0,\Omega}\Big(\sum_\sigma\mathcal{A}_\sigma\Big)^{1/2}\sqrt{2}\Big(\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v^\beta_{\sigma h}\right\|^2_{0,\Omega}\left\|v^\beta_{\sigma h}\right\|^2_{0,\infty,\Omega}\Big)^{1/2}\\
&\le C\left(\left\|v^\beta_{\sigma h}\right\|_{0,\infty,\Omega},\left\|v^\beta_{\sigma h}\right\|_{0,\Omega},C_\star\right)h^{\min(k,k')}. \qquad (5.11)
\end{aligned}
$$

Gathering estimates (5.7)–(5.11), equation (5.6) becomes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{6}\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v^\beta_{\sigma h}\right\|^3_{0,\Omega}+\frac{1}{2}\int_\Omega\left(a_0|\nabla_\perp\phi_h|^2+b_0|\phi_h-\bar{\phi}_h|^2\right)\mathrm{d}x\right)\\
&\quad+\sum_{\beta,\sigma}\mathcal{A}_\sigma\int_{\mathcal{E}_h}\left\{\frac{\beta}{2}\left[\!\left[v^\beta_{\sigma h}\right]\!\right]^2\left(\{\!\{v^\beta_{\sigma h}\}\!\}(\mathcal{D}_{\perp,e}+\mathcal{D}^\beta_{\parallel\sigma,e})-\frac{1}{6}v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\left[\!\left[v^\beta_{\sigma h}\right]\!\right]_\perp\right)\right\}\mathrm{d}\gamma\\
&\quad+\sum_{\beta,\sigma}\mathcal{A}_\sigma\int_{\mathcal{E}_{h_\parallel}}\beta\left[\!\left[\mathcal{R}_h\mathcal{J}_\perp\phi_h\right]\!\right]_\parallel\left[\!\left[v^\beta_{\sigma h}\right]\!\right]_\parallel\left(\frac{1}{2}\mathcal{D}^\beta_{\parallel\sigma,e}+(\alpha-1/2)\{\!\{v^\beta_{\sigma h}\}\!\}\right)\mathrm{d}\gamma\\
&=\mathcal{O}\left(h^{\min(k,k')}\right)\le C_{\mathcal{J}_\perp}h^{\min(k,k')}, \qquad (5.12)
\end{aligned}
$$

where the constant $C_{\mathcal{J}_\perp}=0$ when $\mathcal{J}_\perp=I$ (since the right-hand side of (5.6) vanishes when $\mathcal{J}_\perp=I$, i.e., for the drift-kinetic case) and $C_{\mathcal{J}_\perp}>0$ otherwise (gyrokinetic case). Let us note that the second term of the left-hand side of (5.12) is non-negative since we obviously have

$$
\pm\left(\{\!\{v^\pm_{\sigma h}\}\!\}\mathcal{D}_{\perp,e}-\frac{1}{6}v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot\left[\!\left[v^\pm_{\sigma h}\right]\!\right]_\perp\right)\ge 0 \quad\text{and}\quad \pm\{\!\{v^\pm_{\sigma h}\}\!\}\mathcal{D}^\pm_{\parallel\sigma,e}\ge 0, \qquad (5.13)
$$

as long as $\pm v^{\pm}_{\sigma h} > 0$ (see Corollary 4.15). Now we distinguish the case where $\phi_h \in \widehat{\Phi}^{k'}_h$ or $\phi_h \in \widehat{\Phi}^{k'}_h$. Let us start with the more straightforward case.

Case $\phi_h \in \widehat{\Phi}^{k'}_h$. Since $\phi_h \in \widehat{\Phi}^{k'}_h$, we get $[\![\mathcal{R}_h\mathcal{J}_\perp\phi_h]\!]_{\parallel} = 0$; hence, the third term of the left-hand side of (5.12) vanishes. Therefore from (5.12) and (5.13), we obtain (5.1) and Theorem 5.1 is proved.

Case $\phi_h \in \widehat{\Phi}^{k'}_h$. If we can easily show that the term in parentheses in the third term of the left-hand side of (5.12) remains non-negative, the double bracket terms (in the third term of the left-hand side of (5.12)) have undetermined sign. Therefore, for general Lax–Friedrichs parallel flux (3.11), we cannot say anything about the sign of this term, except in the case where $\alpha = 1/2$ and $\mathcal{D}^{\beta}_{\parallel\sigma,e} = \mathcal{D}_{\parallel,e} = \text{constant} > 0$, i.e., uniform with respect to σ, β and $x_\perp$. If we denote the third term of the left-hand side of (5.12) by R, using the property $\int_{\Omega_\perp} \varphi\mathcal{J}_\perp\psi\,\mathrm{d}x_\perp = \int_{\Omega_\perp} \psi\mathcal{J}_\perp\varphi\,\mathrm{d}x_\perp$, we obtain

$$\begin{aligned}
R &= \int_{\mathcal{E}_{h\parallel}} \mathcal{D}_{\parallel,e}[\![\mathcal{R}_h\mathcal{J}_\perp\phi_h]\!]_{\parallel}[\![\rho_h]\!]_{\parallel}\,\mathrm{d}\gamma \\
&= \int_{\mathcal{E}_{h\parallel}} \mathcal{D}_{\parallel,e}[\![\mathcal{J}_\perp\phi_h]\!]_{\parallel}[\![\rho_h]\!]_{\parallel}\,\mathrm{d}\gamma + \int_{\mathcal{E}_{h\parallel}} \mathcal{D}_{\parallel,e}[\![(\mathcal{R}_h - I)\mathcal{J}_\perp\phi_h]\!]_{\parallel}[\![\rho_h]\!]_{\parallel}\,\mathrm{d}\gamma \\
&= \int_{\mathcal{E}_{h\parallel}} \mathcal{D}_{\parallel,e}[\![\phi_h]\!]_{\parallel}[\![\mathcal{J}_\perp\rho_h]\!]_{\parallel}\,\mathrm{d}\gamma + \int_{\mathcal{E}_{h\parallel}} \mathcal{D}_{\parallel,e}[\![(\mathcal{R}_h - I)\mathcal{J}_\perp\phi_h]\!]_{\parallel}[\![\rho_h]\!]_{\parallel}\,\mathrm{d}\gamma \\
&= \int_{\mathcal{E}_{h\parallel}} \mathcal{D}_{\parallel,e}[\![\phi_h]\!]_{\parallel}[\![\Pi_h\mathcal{J}_\perp\rho_h]\!]_{\parallel}\,\mathrm{d}\gamma \\
&\quad + \int_{\mathcal{E}_{h\parallel}} \mathcal{D}_{\parallel,e}[\![\phi_h]\!]_{\parallel}[\![(I-\Pi_h)\mathcal{J}_\perp\rho_h]\!]_{\parallel}\,\mathrm{d}\gamma + \int_{\mathcal{E}_{h\parallel}} \mathcal{D}_{\parallel,e}[\![(\mathcal{R}_h - I)\mathcal{J}_\perp\phi_h]\!]_{\parallel}[\![\rho_h]\!]_{\parallel}\,\mathrm{d}\gamma \\
&= R_1 + R_2 + R_3.
\end{aligned} \tag{5.14}$$

Using inverse inequalities, stability and high-order approximation properties for the projection-interpolation operators (Section 3.1.3), stability properties of the gyroaverage operator (Section 3.1.4), Lemma 4.14, Corollary 4.16 and Theorem 4.1, we obtain for the term R_2,

$$\begin{aligned}
R_2 &\leq \sum_{e\in\mathcal{E}_{h\parallel}} \mathcal{D}_{\parallel,e}\left\|[\![\phi_h]\!]_{\parallel}\right\|_{0,e}\left\|[\![(I-\Pi_h)\mathcal{J}_\perp\rho_h]\!]_{\parallel}\right\|_{0,e} \\
&\leq \sup_{e\in\mathcal{E}_{h\parallel}} \mathcal{D}_{\parallel,e}\sum_{K\in\mathcal{M}_h} h^{-1}\|\phi_h\|_{0,K}\,\|(I-\Pi_h)\mathcal{J}_\perp\rho_h\|_{0,K} \\
&\leq \sup_{e\in\mathcal{E}_{h\parallel}} \mathcal{D}_{\parallel,e}\|\phi_h\|_{0,\Omega}\left(\|(I-\Pi_h)\mathcal{J}_\perp\rho\|_{0,\Omega} + \|(I-\Pi_h)\mathcal{J}_\perp[\rho-\rho_h]\|_{0,\Omega}\right) \\
&\leq C\left(\sup_{e\in\mathcal{E}_{h\parallel}} \mathcal{D}_{\parallel,e}, \|\phi_h\|_{0,\Omega}, \|\rho\|_{k+1,\Omega}, C_\star\right)h^{\min(k,k')},
\end{aligned} \tag{5.15}$$

while the term R_3 is bounded as

$$
\begin{aligned}
R_3 &\leq \sum_{e\in\mathcal{E}_{h_\parallel}} \mathcal{D}_{\parallel,e} \left\| [\![\rho_h]\!]_{\parallel} \right\|_{0,e} \left\| [\![(I-\mathcal{R}_h)\mathcal{J}_\perp\phi_h]\!]_{\parallel} \right\|_{0,e} \\
&\leq \sup_{e\in\mathcal{E}_{h_\parallel}} \mathcal{D}_{\parallel,e} \sum_{K\in\mathcal{M}_h} h^{-1} \|\rho_h\|_{0,K} \, \|(I-\mathcal{R}_h)\mathcal{J}_\perp\phi_h\|_{0,K} \\
&\leq \sup_{e\in\mathcal{E}_{h_\parallel}} \mathcal{D}_{\parallel,e} \|\rho_h\|_{0,\Omega} \left(\|(I-\mathcal{R}_h)\mathcal{J}_\perp\phi\|_{0,\Omega} + \|(I-\mathcal{R}_h)\mathcal{J}_\perp[\phi-\phi_h]\|_{0,\Omega} \right) \\
&\leq C\left(\sup_{e\in\mathcal{E}_{h_\parallel}} \mathcal{D}_{\parallel,e}, \|\rho_h\|_{0,\Omega}\,, \|\phi\|_{k'+1,\Omega}, C_\star \right) h^{\min(k,k')}. \qquad (5.16)
\end{aligned}
$$

It remains to deal with the term R_1. From the construction of the finite element spaces (i.e., from the Cartesian product between parallel and perpendicular directions and the discontinuous polynomial basis in the parallel direction) and (3.14), where we take $\phi_h = \psi_{h\perp} \otimes \psi_{h\parallel}$, with $\psi_{h\perp} = \mathcal{D}_{\parallel,e} [\![\phi_h]\!]_{\parallel}$, we obtain

$$
\int_{\mathcal{E}_{h_\parallel}} \mathcal{D}_{\parallel,e} [\![\phi_h]\!]_{\parallel} [\![\Pi_h\mathcal{J}_\perp\rho_h]\!]_{\parallel} d\gamma = \int_{\mathcal{E}_{h_\parallel}} \mathcal{D}_{\parallel,e} \left(a_0 \left|\nabla_\perp [\![\phi_h]\!]_{\parallel}\right|^2 + b_0 \left|[\![\phi_h]\!]_{\parallel}\right|^2 \right) d\gamma. \qquad (5.17)
$$

Gathering (5.14)–(5.17), expression (5.12) becomes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{6}\sum_{\beta,\sigma} \mathcal{A}_\sigma \left\| v_{\sigma h}^{\beta} \right\|_{0,\Omega}^{3} + \frac{1}{2}\int_\Omega \left(a_0|\nabla_\perp\phi_h|^2 + b_0|\phi_h-\bar{\phi}_h|^2 \right) \mathrm{d}x \right) \\
&\quad + \sum_{\beta,\sigma} \mathcal{A}_\sigma \int_{\mathcal{E}_h} \left\{ \frac{\beta}{2} [\![v_{\sigma h}^{\beta}]\!]^2 \left(\{\!\{v_{\sigma h}^{\beta}\}\!\}(\mathcal{D}_{\perp,e} + \mathcal{D}_{\parallel\sigma,e}^{\beta}) - \frac{1}{6} v_E(\mathcal{R}_h\mathcal{J}_\perp\phi_h)\cdot [\![v_{\sigma h}^{\beta}]\!]_\perp \right) \right\} \mathrm{d}\gamma \\
&\quad + \int_{\mathcal{E}_{h_\parallel}} \mathcal{D}_{\parallel,e} \left(a_0 \left|\nabla_\perp [\![\phi_h]\!]_{\parallel}\right|^2 + b_0 \left|[\![\phi_h]\!]_{\parallel}\right|^2 \right) \mathrm{d}\gamma \\
&= \mathcal{O}\left(h^{\min(k,k')} \right) \leq C_{\mathcal{J}_\perp} h^{\min(k,k')}. \qquad (5.18)
\end{aligned}
$$

Since the second and third terms of the left-hand side of (5.18) are non-negative, we obtain (5.1) and Theorem 5.1 is proved. □

6. Conclusion and future developments

In this article, we have designed discontinuous Galerkin schemes for the gyrokinetic-waterbag equations. We have proved the convergence and high-order *a priori* error estimates for such schemes. We plan to implement this scheme and to make comparisons with other schemes and codes in forthcoming papers. First, we intend to compare SL schemes (Besse & Bertrand, 2009; Coulette & Besse, 2013a,b) and such DGFEM schemes to understand how SL schemes behave for entropic weak solutions of the gyrokinetic-waterbag equations. Second, we plan to compare entropy-dissipating weak solutions, given by these present DGFEM schemes and entropy-preserving solutions of the gyrokinetic-waterbag equations. Entropy-preserving solutions can be obtained from a pure Lagrangian scheme, which consists in following the Lagrangian evolution (with possible folding) of three-dimensional manifolds (contours), embedded in a four-dimensional phase space. Eventually, we could compare the DGFEM schemes (for the gyrokinetic-waterbag equations) to gyrokinetic-Vlasov codes.

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Appendix A. Proofs of the error estimates for the electrical potential

In this appendix, we give the proof of Proposition 4.3 (respectively Proposition 4.4) related to L^2-estimates (respectively L^∞-estimates) of the electrical potential and its gradient. For this purpose, we introduce the variational problem given by the weak formulation of the quasi-neutrality equation (2.9). It consists in finding $\bar{\phi} \in H_0^1(\Omega_\perp)$ and $\phi \in L^2\left(\Omega_\parallel;\ H_0^1(\Omega_\perp)\right)$ such that

$$a_\perp(\bar{\phi}, \bar{\psi}) = L_\perp(\bar{\psi}) \quad \forall \bar{\psi} \in H_0^1(\Omega_\perp) \tag{A.1}$$

and

$$a(\phi, \psi) = L(\psi) \quad \forall \psi \in L^2\left(\Omega_\parallel;\ H_0^1(\Omega_\perp)\right). \tag{A.2}$$

The bilinear forms $a_\perp(\cdot,\cdot)$ and $a(\cdot,\cdot)$ have been defined in Section 3.2.2, and the linear forms $L_\perp(\cdot)$ and $L(\cdot)$ are given by

$$L_\perp(\bar\psi) = \int_{\Omega_\perp} \bar\rho\bar\psi\,\mathrm{d}x \quad \forall\bar\psi \in H_0^1(\Omega_\perp) \quad \text{and} \quad L(\psi) = \int_\Omega \tilde\rho\psi\,\mathrm{d}x \quad \forall\psi \in L^2\left(\Omega_\parallel;\, H_0^1(\Omega_\perp)\right),$$

with

$$\rho = \sum_{\sigma\in\mathfrak{L}^{\mathrm{NM}}} \mathcal{A}_\sigma(v_\sigma^+ - v_\sigma^-), \qquad \bar\rho = \langle\mathcal{J}_\perp\rho\rangle_\parallel - n_{i0} = \frac{1}{|\Omega_\parallel|}\int_{\Omega_\parallel} \mathcal{J}_\perp\rho\,\mathrm{d}x_\parallel - n_{i0} \quad \text{and} \quad \tilde\rho = \mathcal{J}_\perp\rho + b_0\bar\phi - n_{i0}.$$

Let us note that on the one hand the bilinear forms $a_\perp(\cdot,\cdot)$ and $a(\cdot,\cdot)$ are, respectively, elliptic and continuous in $H_0^1(\Omega_\perp)$ and $L^2\left(\Omega_\parallel;\, H_0^1(\Omega_\perp)\right)$, and on the other hand, the linear forms $L_\perp(\cdot)$ and $L(\cdot)$ are, respectively, continuous in $H_0^1(\Omega_\perp)$ and $L^2\left(\Omega_\parallel;\, H_0^1(\Omega_\perp)\right)$, as long as $v_\sigma^\pm \in L^2(\Omega)$ for all $\sigma \in \mathfrak{L}^{\mathrm{NM}}$. Hence, using the Lax–Milgram theorem, problem (A.1) (respectively (A.2)) has a unique solution $\bar\phi \in H_0^1(\Omega_\perp)$ (respectively $\phi \in L^2\left(\Omega_\parallel;\, H_0^1(\Omega_\perp)\right)$).

A.1 *Proof of Proposition 4.3*

A.1.1 *$H^1(\Omega_\perp)$-error estimates for $\bar\phi_h$.* The first part of the proof of Proposition 4.3 is to show the estimate

$$\|\bar\phi(t) - \bar\phi_h(t)\|_{1,\Omega_\perp} \leq C_2 h^{\min(k+1,k')} + C_2\left(\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v_\sigma^\beta(t) - v_{\sigma h}^\beta(t)\right\|_{0,\Omega}^2\right)^{1/2}, \tag{A.3}$$

where the constant C_2 is the constant involved in Proposition 4.3.

Let us first estimate the term $\|\bar\rho - \bar\rho_h\|_{0,\Omega_\perp}$. Using the Cauchy–Schwarz inequality, L^2-stability and the approximation property of the operator Π_h (Section 3.1.3) and the H^m-stability property of the operator $\mathcal{J}_\perp$ (Section 3.1.4), we then obtain

$$\begin{aligned}
\|\bar\rho - \bar\rho_h\|_{0,\Omega_\perp}^2 &= \frac{1}{|\Omega_\parallel|^2}\int_{\Omega_\perp} dx_\perp\left(\int_{\Omega_\perp} dx_\parallel \sum_{\beta,\sigma}\beta\mathcal{A}_\sigma\left(\Pi_h\mathcal{J}_\perp v_{\sigma h}^\beta - \mathcal{J}_\perp v_\sigma^\beta\right)\right)^2 \\
&= \frac{1}{|\Omega_\parallel|^2}\int_{\Omega_\perp} dx_\perp\left(\int_{\Omega_\perp} dx_\parallel \sum_{\beta,\sigma}\beta\mathcal{A}_\sigma\left\{\Pi_h\mathcal{J}_\perp\left(v_{\sigma h}^\beta - v_\sigma^\beta\right) + (\Pi_h - I)\mathcal{J}_\perp v_\sigma^\beta\right\}\right)^2 \\
&\leq \frac{4\sum_\sigma\mathcal{A}_\sigma}{|\Omega_\parallel|}\sum_{\beta,\sigma}\mathcal{A}_\sigma\left(\left\|\Pi_h\mathcal{J}_\perp\left(v_{\sigma h}^\beta - v_\sigma^\beta\right)\right\|_{0,\Omega}^2 + \left\|(\Pi_h - I)\mathcal{J}_\perp v_\sigma^\beta\right\|_{0,\Omega}^2\right) \\
&\leq \frac{4C\sum_\sigma\mathcal{A}_\sigma}{|\Omega_\parallel|}\sum_{\beta,\sigma}\mathcal{A}_\sigma\left(\left\|v_{\sigma h}^\beta - v_\sigma^\beta\right\|_{0,\Omega}^2 + h^{2k+2}\left\|v_\sigma^\beta\right\|_{k+1,\Omega}^2\right) \\
&\leq Ch^{2k+2} + \sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v_\sigma^\beta - v_{\sigma h}^\beta\right\|_{0,\Omega}^2.
\end{aligned} \tag{A.4}$$

We then estimate the term $\|\bar{\phi}-\bar{\phi}_h\|_{1,\Omega_\perp}$. For any $\bar{\chi}_h \in \bar{\Phi}^{k'}_{h_\perp}$, using the approximation properties of the space $\bar{\Phi}^{k'}_{h_\perp}$ (Section 3.1.3), we have

$$\begin{aligned}\|\bar{\phi}-\bar{\phi}_h\|_{1,\Omega_\perp} &\le \|\bar{\phi}-\bar{\chi}_h\|_{1,\Omega_\perp}+\|\bar{\chi}_h-\bar{\phi}_h\|_{1,\Omega_\perp}\\ &\le Ch_\perp^{k'}\|\bar{\phi}\|_{k'+1,\Omega_\perp}+\|\bar{\chi}_h-\bar{\phi}_h\|_{1,\Omega_\perp}\\ &\le C|\Omega_\parallel|^{-1/2}h^{k'}\|\phi\|_{k'+1,\Omega}+\|\bar{\chi}_h-\bar{\phi}_h\|_{1,\Omega_\perp}.\end{aligned} \tag{A.5}$$

Since the bilinear form $a_\perp(\cdot,\cdot)$ is H^1_0-elliptic with a coercivity constant $C^\perp_o = C^\perp_o(a_0)$ and H^1_0-continuous with a stability constant $C^\perp_s = C^\perp_s(a_0)$, then using (A.1) and (3.13), we obtain

$$\begin{aligned}\|\bar{\chi}_h-\bar{\phi}_h\|^2_{1,\Omega_\perp} &\le \frac{1}{C^\perp_o}a_\perp(\bar{\phi}_h-\bar{\chi}_h,\bar{\phi}_h-\bar{\chi}_h)\\ &\le \frac{1}{C^\perp_o}a_\perp(\bar{\phi}-\bar{\chi}_h,\bar{\phi}_h-\bar{\chi}_h)+\frac{1}{C^\perp_o}\left(L_{\perp h}(\bar{\phi}_h-\bar{\chi}_h)-L_\perp(\bar{\phi}_h-\bar{\chi}_h)\right)\\ &\le \frac{C^\perp_s}{C^\perp_o}\|\bar{\phi}-\bar{\chi}_h\|_{1,\Omega_\perp}\|\bar{\phi}_h-\bar{\chi}_h\|_{1,\Omega_\perp}+\frac{1}{C^\perp_o}\left(L_{\perp h}(\bar{\phi}_h-\bar{\chi}_h)-L_\perp(\bar{\phi}_h-\bar{\chi}_h)\right),\end{aligned}$$

from which we infer that

$$\|\bar{\chi}_h-\bar{\phi}_h\|_{1,\Omega_\perp} \le C|\Omega_\parallel|^{-1/2}\frac{C^\perp_s}{C^\perp_o}h^{k'}\|\phi\|_{k'+1,\Omega}+\sup_{\bar{w}_h\in\bar{\Phi}^{k'}_{h_\perp}}\frac{|L_{\perp h}(\bar{w}_h)-L_\perp(\bar{w}_h)|}{\|\bar{w}_h\|_{1,\Omega_\perp}}. \tag{A.6}$$

In (A.6) we have used the approximation properties of the space $\bar{\Phi}^{k'}_{h_\perp}$ (Section 3.1.3). Using the Cauchy–Schwarz inequality, we obtain

$$|L_{\perp h}(\bar{w}_h)-L_\perp(\bar{w}_h)| \le \|\bar{w}_h\|_{1,\Omega_\perp}\|\bar{\rho}-\bar{\rho}_h\|_{0,\Omega_\perp}.$$

From the previous inequality and (A.4)–(A.6), we obtain the H^1-estimate (A.3).

A.1.2 *$L^2(\Omega_\perp)$-error estimates for $\bar{\phi}_h$.* We now establish the L^2-norm error estimate of the parallel-averaged electrical potential, which is given by the inequality

$$\|\bar{\phi}(t)-\bar{\phi}_h(t)\|_{0,\Omega_\perp} \le C_2h^{\min(k,k')+1}+C_2\left(\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v^\beta_\sigma(t)-v^\beta_{\sigma h}(t)\right\|^2_{0,\Omega}\right)^{1/2}. \tag{A.7}$$

The constant C_2 is the constant involved in Proposition 4.3. To obtain (A.7), we introduce the following dual problem. Let $\bar{g} \in L^2(\Omega_\perp)$ and $\bar{\varphi}_{\bar{g}} \in H^1_0(\Omega_\perp)$ be the unique solution (by the Lax–Milgram theorem) of the dual variational problem

$$a_\perp(\bar{v},\bar{\varphi}_{\bar{g}}) = \langle\bar{g},\bar{v}\rangle_{\Omega_\perp} = \int_{\Omega_\perp}\bar{v}\bar{g}\,\mathrm{d}x_\perp \quad \forall\bar{v}\in H^1_0(\Omega_\perp).$$

Moreover, from regularity results for elliptic partial differential equations of second order (see Gilbarg & Trudinger, 1998), we know that $\bar{\varphi}_{\bar{g}} \in H^2(\Omega_\perp)$ and that the following inequality holds:

$$\|\bar{\varphi}_{\bar{g}}\|_{2,\Omega_\perp} \leq C(a_0, \Omega_\perp)\|\bar{g}\|_{0,\Omega_\perp}. \tag{A.8}$$

Since, for all $\bar{\psi}_h \in \bar{\Phi}^{k'}_{h_\perp}$ we have $a_\perp(\bar{\phi}_h, \bar{\psi}_h) = \langle \bar{\rho}_h, \bar{\psi}_h\rangle_{\Omega_\perp}$ and $a_\perp(\bar{\phi}, \bar{\psi}_h) = \langle \bar{\rho}, \bar{\psi}_h\rangle_{\Omega_\perp}$, we then get

$$a_\perp(\bar{\phi} - \bar{\phi}_h, \bar{\varphi}_h) = L_\perp(\bar{\varphi}_h) - L_{\perp h}(\bar{\varphi}_h) = \langle \bar{\rho} - \bar{\rho}_h, \bar{\varphi}_h\rangle_{\Omega_\perp}, \quad \bar{\varphi}_h \in \bar{\Phi}^{k'}_{h_\perp}.$$

Since $(\bar{\phi} - \bar{\phi}_h) \in H^1_0(\Omega_\perp)$, using the previous equality and approximation properties of the space $\bar{\Phi}^{k'}_{h_\perp}$ (Section 3.1.3), we find

$$\begin{aligned}
\langle \bar{g}, \bar{\phi} - \bar{\phi}_h\rangle_{\Omega_\perp} &= a_\perp(\bar{\phi} - \bar{\phi}_h, \bar{\varphi}_{\bar{g}}) \\
&= a_\perp(\bar{\phi} - \bar{\phi}_h, \bar{\varphi}_{\bar{g}} - \bar{\varphi}_h) + L_\perp(\bar{\varphi}_h) - L_{\perp h}(\bar{\varphi}_h) \\
&\leq \inf_{\bar{\varphi}_h \in \bar{\Phi}^{k'}_{h_\perp}} \left\{ C^\perp_s \|\bar{\varphi}_{\bar{g}} - \bar{\varphi}_h\|_{1,\Omega_\perp} \|\bar{\phi} - \bar{\phi}_h\|_{1,\Omega_\perp} + |L_\perp(\bar{\varphi}_h) - L_{\perp h}(\bar{\varphi}_h)| \right\} \\
&\leq C^\perp_s Ch \|\bar{\varphi}_{\bar{g}}\|_{2,\Omega_\perp} \|\bar{\phi} - \bar{\phi}_h\|_{1,\Omega_\perp} + \inf_{\bar{\varphi}_h \in \bar{\Phi}^{k'}_{h_\perp}} |L_\perp(\bar{\varphi}_h) - L_{\perp h}(\bar{\varphi}_h)|,
\end{aligned}$$

from which, using (A.8) we infer

$$\begin{aligned}
\|\bar{\phi} - \bar{\phi}_h\|_{0,\Omega_\perp} &= \sup_{\bar{g} \in L^2(\Omega_\perp)} \frac{\langle \bar{g}, \bar{\phi} - \bar{\phi}_h\rangle_{\Omega_\perp}}{\|\bar{g}\|_{0,\Omega_\perp}} \\
&\leq Ch\|\bar{\phi} - \bar{\phi}_h\|_{1,\Omega_\perp} + \sup_{\bar{g} \in L^2(\Omega_\perp)} \inf_{\bar{\varphi}_h \in \bar{\Phi}^{k'}_{h_\perp}} \frac{|L_\perp(\bar{\varphi}_h) - L_{\perp h}(\bar{\varphi}_h)|}{\|\bar{g}\|_{0,\Omega_\perp}}.
\end{aligned} \tag{A.9}$$

Now by taking $\bar{\varphi}_h = \mathbb{I}_{h_\perp} \bar{\varphi}_{\bar{g}}$ and using the inequality (see Ciarlet, 1991, inequality (17.6))

$$\left(\sum_{K_\perp \in \mathcal{T}_{h_\perp}} \|\mathbb{I}_{h_\perp} \bar{\varphi}_{\bar{g}}\|^2_{2,K_\perp} \right)^{1/2} \leq C\|\bar{\varphi}_{\bar{g}}\|_{2,\Omega_\perp}, \tag{A.10}$$

we obtain

$$\begin{aligned}
|L_\perp(\bar{\varphi}_h) - L_{\perp h}(\bar{\varphi}_h)| &\leq \|\bar{\rho} - \bar{\rho}_h\|_{0,\Omega_\perp} \|\bar{\varphi}_h\|_{0,\Omega_\perp} \leq \|\bar{\rho} - \bar{\rho}_h\|_{0,\Omega_\perp} \left(\sum_{K_\perp \in \mathcal{T}_{h_\perp}} \|\bar{\varphi}_h\|^2_{2,K_\perp} \right)^{1/2} \\
&\leq C\|\bar{\rho} - \bar{\rho}_h\|_{0,\Omega_\perp} \|\bar{\varphi}_{\bar{g}}\|_{2,\Omega_\perp},
\end{aligned}$$

from which, using (A.8)–(A.9), we infer that

$$\|\bar{\phi} - \bar{\phi}_h\|_{0,\Omega_\perp} \leq C \left(h\|\bar{\phi} - \bar{\phi}_h\|_{1,\Omega_\perp} + \|\bar{\rho} - \bar{\rho}_h\|_{0,\Omega_\perp} \right).$$

Estimates (A.3)–(A.4) and the previous inequality lead to (A.7) and complete the proof.

A.1.3 $L^2\left(\Omega_{\|};\, H^1(\Omega_\perp)\right)$*-error estimate for* ϕ_h. Let us first estimate the term $\|\tilde\rho - \tilde\rho_h\|_{0,\Omega}$. Using the Cauchy–Schwarz inequality, L^2-stability and the approximation property of the operator Π_h (Section 3.1.3), the H^m-stability property of the operator $\mathcal{J}_\perp$ (Section 3.1.4) and (A.7), we then obtain

$$\begin{aligned}\|\tilde\rho - \tilde\rho_h\|_{0,\Omega} &\le \|\Pi_h \mathcal{J}_\perp(\rho-\rho_h)\|_{0,\Omega} + \|(\Pi_h - I)\mathcal{J}_\perp\rho\|_{0,\Omega} + L_{\|}^{1/2}\|b_0\|_{0,\infty,\Omega_\perp}\|\bar\phi - \bar\phi_h\|_{0,\Omega_\perp}\\ &\le C\|\rho-\rho_h\|_{0,\Omega} + Ch^{k+1}\|\rho\|_{k+1,\Omega} + L_{\|}^{1/2}\|b_0\|_{0,\infty,\Omega_\perp}\|\bar\phi - \bar\phi_h\|_{0,\Omega_\perp}\\ &\le Ch^{\min(k,k')+1} + C\left(\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v_\sigma^\beta - v_{\sigma h}^\beta\right\|_{0,\Omega}^2\right)^{1/2}. \end{aligned}\tag{A.11}$$

Now for any $\chi_h \in \Phi_h^{k'}$, using approximation properties of the space $\Phi_h^{k'}$ (Section 3.1.3), we have

$$\begin{aligned}\|\phi-\phi_h\|_{L^2\left(\Omega_{\|};H_0^1(\Omega_\perp)\right)} &\le \|\phi-\chi_h\|_{L^2\left(\Omega_{\|};H_0^1(\Omega_\perp)\right)} + \|\chi_h-\phi_h\|_{L^2\left(\Omega_{\|};H_0^1(\Omega_\perp)\right)}\\ &\le Ch^{k'}\|\phi\|_{k'+1,\Omega} + \|\chi_h-\phi_h\|_{L^2\left(\Omega_{\|};H_0^1(\Omega_\perp)\right)}. \end{aligned}\tag{A.12}$$

Since the bilinear form $a(\cdot,\cdot)$ is $L^2\left(\Omega_{\|};\, H_0^1(\Omega_\perp)\right)$-elliptic with a coercivity constant $C_o = C_o(a_0, b_0)$ and $L^2\left(\Omega_{\|};\, H_0^1(\Omega_\perp)\right)$-continuous with a stability constant $C_s = C_s(a_0, b_0)$, then using (A.2) and (3.14), we obtain

$$\begin{aligned}\|\chi_h-\phi_h\|^2_{L^2\left(\Omega_{\|};H_0^1(\Omega_\perp)\right)} &\le \frac{1}{C_o}a(\phi_h-\chi_h,\phi_h-\chi_h)\\ &\le \frac{1}{C_o}a(\phi-\chi_h,\phi_h-\chi_h) + \frac{1}{C_o}\left(L_h(\phi_h-\chi_h) - L(\phi_h-\chi_h)\right)\\ &\le \frac{C_s}{C_o}\|\phi-\chi_h\|_{L^2\left(\Omega_{\|};H_0^1(\Omega_\perp)\right)}\|\phi_h-\chi_h\|_{L^2\left(\Omega_{\|};H_0^1(\Omega_\perp)\right)}\\ &\quad + \frac{1}{C_o}\left(L_h(\phi_h-\chi_h) - L(\phi_h-\chi_h)\right), \end{aligned}$$

from which we infer

$$\|\chi_h-\phi_h\|_{L^2\left(\Omega_{\|};H_0^1(\Omega_\perp)\right)} \le Ch^{k'}\|\phi\|_{k'+1,\Omega} + \sup_{w_h\in\Phi_h^{k'}}\frac{|L_{\perp h}(\bar w_h) - L_\perp(\bar w_h)|}{\|w_h\|_{L^2\left(\Omega_{\|};H_0^1(\Omega_\perp)\right)}}.\tag{A.13}$$

In (A.13) we have used approximation properties of the space $\Phi_h^{k'}$ (Section 3.1.3). Now using the Cauchy–Schwarz inequality, we obtain

$$|L_h(w_h) - L(w_h)| \le \|w_h\|_{L^2\left(\Omega_{\|};H_0^1(\Omega_\perp)\right)}\|\tilde\rho - \tilde\rho_h\|_{0,\Omega_\perp}.$$

By (A.11)–(A.13) and from the previous equality, we obtain (4.1) with $|\alpha| = 1$.

A.1.4 *$L^2(\Omega)$-error estimate for ϕ_h.* We now aim to estimate the term $\|\phi - \phi_h\|_{0,\Omega}$, i.e.,

$$\|\phi - \phi_h\|_{0,\Omega} = \sup_{\substack{f \in L^2(\Omega) \\ f \neq 0}} \frac{\langle f, \phi - \phi_h \rangle_\Omega}{\|f\|_{0,\Omega}}.$$

Since $f \in L^2(\Omega)$, there exists a sequence $\{f^n\}_{n \geq 0}$ in $L^2(\Omega)$ such that $f^n = f_\perp^n(x_\perp) \otimes f_\parallel^n(x_\parallel)$, with $f_\perp^n(x_\perp) \in L^2(\Omega_\perp)$ and $f_\parallel^n(x_\parallel) \in L^2(\Omega_\parallel)$, and such that

$$\forall \varepsilon > 0, \ \exists n_0 \ \text{ such that } \ n > n_0 \implies \|f^n - f\|_{0,\Omega} \leq \varepsilon.$$

Without loss of generality, we take the sequence $\{f^n\}_{n \geq 0}$ such that $\|f^n\|_{0,\Omega} = \|f\|_{0,\Omega} = 1$, for all $n \geq 0$. Therefore, we have

$$\begin{aligned} \|\phi - \phi_h\|_{0,\Omega} &\leq |\langle f - f^n, \phi - \phi_h \rangle_\Omega| + |\langle f^n, \phi - \phi_h \rangle_\Omega| \\ &\leq \varepsilon \|\phi - \phi_h\|_{0,\Omega} + \langle f^n, \phi - \phi_h \rangle_\Omega. \end{aligned} \tag{A.14}$$

Then it remains to estimate the second term of the right-hand side of (A.14). For this purpose, we introduce the following second-order elliptic problem with Neumann boundary conditions. Let $\varphi_{f_\perp^n} \in H^2(\Omega_\perp)$ be the unique solution of the problem

$$\begin{cases} \mathcal{L}\varphi_{f_\perp^n} = f_\perp^n \ \text{ in } \ \Omega_\perp, \\ \dfrac{\partial \varphi_{f_\perp^n}}{\partial \nu_\perp} = 0 \ \text{ on } \ \partial\Omega_\perp, \end{cases}$$

where

$$\mathcal{L}\psi = -\nabla_\perp \cdot (a_0 \nabla_\perp \psi) + b_0 \psi \quad \forall \psi \in H^2(\Omega_\perp).$$

From regularity results for elliptic partial differential equations of second order (see Gilbarg & Trudinger, 1998), the following estimate holds:

$$\|\varphi_{f_\perp^n}\|_{2,\Omega_\perp} \leq C(a_0, b_0, \Omega_\perp) \|f_\perp^n\|_{0,\Omega_\perp}. \tag{A.15}$$

Since $\mathcal{L}$ does not depend on $x_\parallel$, we get

$$f^n = f_\perp^n \otimes f_\parallel^n = \mathcal{L}\varphi_{f_\perp^n} \otimes f_\parallel^n = \mathcal{L}(\varphi_{f_\perp^n} \otimes f_\parallel^n) = \mathcal{L}\varphi_{f^n},$$

where we have set $\varphi_{f^n} = \varphi_{f_\perp^n} \otimes f_\parallel^n$. Therefore, introducing the bilinear form $\tilde{a}(\cdot, \cdot)$ defined by

$$\tilde{a}(\varphi, \psi) = \int_{\Omega_\perp} (a_0 \nabla_\perp \varphi \cdot \nabla_\perp \psi + b_0 \varphi \psi) \, \mathrm{d}x_\perp, \quad \varphi, \psi \in H^1(\Omega_\perp), \tag{A.16}$$

and the quantities

$$\tilde{\phi}_{h\perp}^n = \int_{\Omega_\parallel} \phi_h f_\parallel^n \, \mathrm{d}x_\parallel, \quad \phi_\perp^n = \int_{\Omega_\parallel} \phi f_\parallel^n \, \mathrm{d}x_\parallel,$$

we obtain

$$\begin{aligned}
\langle f^n, \phi - \phi_h \rangle_\Omega &= \langle \mathcal{L}\varphi_{f^n}, \phi - \phi_h \rangle_\Omega \\
&= a(\phi - \phi_h, \varphi_{f^n}) = a\left(\phi - \phi_h, \varphi_{f^n_\perp} \otimes f^n_\parallel\right) = a\left((\phi - \phi_h) f^n_\parallel, \varphi_{f^n_\perp}\right) \\
&= \tilde{a}\left(\phi^n_\perp - \tilde{\phi}^n_{h\perp}, \varphi_{f^n_\perp}\right) \\
&= \tilde{a}\left(\phi^n_\perp - \tilde{\phi}^n_{h\perp}, \varphi_{f^n_\perp} - \mathbb{I}_{h_\perp}\varphi_{f^n_\perp}\right) + \tilde{a}\left(\phi^n_\perp - \tilde{\phi}^n_{h\perp}, \mathbb{I}_{h_\perp}\varphi_{f^n_\perp}\right) \\
&= a\left(\phi - \phi_h, (\varphi_{f^n_\perp} - \mathbb{I}_{h_\perp}\varphi_{f^n_\perp}) \otimes f^n_\parallel\right) + \tilde{a}\left(\phi^n_\perp - \tilde{\phi}^n_{h\perp}, \mathbb{I}_{h_\perp}\varphi_{f^n_\perp}\right). \qquad \text{(A.17)}
\end{aligned}$$

Using the approximation property of interpolation operator $\mathbb{I}_{h_\perp}$ (Section 3.1.3) and regularity estimate (A.15), the first term of the right-hand side of (A.17) can be estimated as

$$\begin{aligned}
a\left(\phi - \phi_h, (\varphi_{f^n_\perp} - \mathbb{I}_{h_\perp}\varphi_{f^n_\perp}) \otimes f^n_\parallel\right) &\le C_s \|\phi - \phi_h\|_{L^2\left(\Omega_\parallel; H^1_0(\Omega_\perp)\right)} \|f^n_\parallel\|_{0,\Omega_\parallel} \|\varphi_{f^n_\perp} - \mathbb{I}_{h_\perp}\varphi_{f^n_\perp}\|_{1,\Omega_\perp} \\
&\le Ch \|\phi - \phi_h\|_{L^2\left(\Omega_\parallel; H^1_0(\Omega_\perp)\right)} \|f^n_\parallel\|_{0,\Omega_\parallel} \|\varphi_{f^n_\perp}\|_{2,\Omega_\perp} \\
&\le Ch \|\phi - \phi_h\|_{L^2\left(\Omega_\parallel; H^1_0(\Omega_\perp)\right)}. \qquad \text{(A.18)}
\end{aligned}$$

Now, the second term of the right-hand side of (A.17) can be decomposed as

$$\tilde{a}\left(\phi^n_\perp - \tilde{\phi}^n_{h\perp}, \mathbb{I}_{h_\perp}\varphi_{f^n_\perp}\right) = \tilde{a}\left(\phi^n_{h\perp} - \tilde{\phi}^n_{h\perp}, \mathbb{I}_{h_\perp}\varphi_{f^n_\perp}\right) + \tilde{a}\left(\phi^n_\perp - \phi^n_{h\perp}, \mathbb{I}_{h_\perp}\varphi_{f^n_\perp}\right), \qquad \text{(A.19)}$$

where $\phi^n_{h\perp} \in H^1_0(\Omega_\perp)$ is the unique solution of the variational problem

$$\tilde{a}(\phi^n_{h\perp}, \psi_{\perp h}) = \tilde{L}^n_{h\perp}(\psi_{\perp h}) \quad \forall \psi_{\perp h} \in \bar{\Phi}^{k'}_{h_\perp} \cap H^1_0(\Omega_\perp), \qquad \text{(A.20)}$$

with

$$\tilde{L}^n_{h\perp}(\psi_\perp) = \int_{\Omega_\perp} \tilde{\rho}^n_{h\perp} \psi_\perp \, \mathrm{d}x_\perp \quad \forall \psi_\perp \in L^2(\Omega_\perp), \quad \tilde{\rho}^n_{h\perp} = \int_{\Omega_\parallel} \tilde{\rho}_h f^n_{\parallel h} \, \mathrm{d}x_\parallel \quad \text{and} \quad f^n_{\parallel h} = \mathbb{P}_{h_\parallel} f^n_\parallel.$$

Let us note that the variational problem (A.20) is obtained from the variational problem (3.14) by taking the test function $\psi_h = \psi_{\perp h} \otimes f^n_{\parallel h}$. Moreover, we observe that the function $\phi^n_\perp \in H^1_0(\Omega_\perp)$ is the unique solution of the variational problem

$$\tilde{a}(\phi^n_\perp, \psi_\perp) = \tilde{L}^n_\perp(\psi_\perp) \quad \forall \psi_\perp \in H^1_0(\Omega_\perp),$$

obtained from the variational problem (A.2) by taking the test function $\psi = \psi_\perp \otimes f^n_\parallel$. Let us deal with the first term of the right-hand side of (A.19). Using the H^2-stability inequality (A.10) and the regularity estimate (A.15), we get

$$
\begin{aligned}
\tilde{a}\left(\phi^n_{h\perp} - \tilde{\phi}^n_{h\perp}, \mathbb{I}_{h_\perp}\varphi_{f^n_\perp}\right) &= \int_\Omega (f^n_\parallel - f^n_{\parallel h})(a_0 \nabla_\perp \phi_h \cdot \nabla_\perp \mathbb{I}_{h_\perp}\varphi_{f^n_\perp} + b_0 \phi_h \mathbb{I}_{h_\perp}\varphi_{f^n_\perp})\,\mathrm{d}x \\
&= \int_\Omega (f^n_\parallel - f^n_{\parallel h})\phi_h \mathcal{L}\,\mathbb{I}_{h_\perp}\varphi_{f^n_\perp}\,\mathrm{d}x \\
&\le C(a_0, b_0)\|\phi_h\|_{0,\Omega}\|f^n_\parallel - f^n_{\parallel h}\|_{0,\Omega_\parallel}\left(\sum_{K_\perp \in \mathcal{T}_{h_\perp}} \|\mathbb{I}_{h_\perp}\varphi_{f^n_\perp}\|^2_{H^2(K_\perp)}\right)^{1/2} \\
&\le C(\|\phi\|_{0,\Omega} + \|\phi - \phi_h\|_{0,\Omega})\|f^n_\parallel - f^n_{\parallel h}\|_{0,\Omega_\parallel}\|\varphi_{f^n_\perp}\|_{H^2(\Omega_\perp)} \\
&\le C(\|\phi\|_{0,\Omega} + \|\phi - \phi_h\|_{0,\Omega})\|f^n_\parallel - f^n_{\parallel h}\|_{0,\Omega_\parallel}\|f^n_\perp\|_{0,\Omega_\perp}. \qquad \text{(A.21)}
\end{aligned}
$$

On the one hand, from the L^2-stability property of the operator $\mathbb{P}_{h_\parallel}$ (Section 3.1.3), we get $\|f^n_{\parallel h}\|_{0,\Omega_\parallel} \le C\|f^n_\parallel\|_{0,\Omega_\parallel}$. On the other hand, from the consistency property of the operator $\mathbb{P}_{h_\parallel}$, we get $f^n_{\parallel h} \longrightarrow f^n_\parallel$ almost everywhere as $h \to 0$. Therefore, using the Lebesgue theorem of dominated convergence, for all $h > 0$, there exists a $\varepsilon_h > 0$ such that

$$
\|f^n_\parallel - f^n_{\parallel h}\|_{0,\Omega_\parallel} \le \varepsilon_h.
$$

Using this previous estimate, (A.21) becomes

$$
\tilde{a}\left(\phi^n_{h\perp} - \tilde{\phi}^n_{h\perp}, \mathbb{I}_{h_\perp}\varphi_{f^n_\perp}\right) \le C\varepsilon_h\|\phi - \phi_h\|_{0,\Omega} + C\|\phi\|_{0,\Omega}. \qquad \text{(A.22)}
$$

Taking $\phi - \chi_h$ instead of ϕ, where $\chi_h \in \Phi^{k'}_h$ and using approximation properties in the space $\Phi^{k'}_h$, from (A.22) we obtain

$$
\begin{aligned}
\tilde{a}\left(\phi^n_{h\perp} - \tilde{\phi}^n_{h\perp}, \mathbb{I}_{h_\perp}\varphi_{f^n_\perp}\right) &\le C\varepsilon_h\|\phi - \phi_h\|_{0,\Omega} + C \inf_{\chi_h \in \Phi^{k'}_h}\|\phi - \chi_h\|_{0,\Omega} \\
&\le C\varepsilon_h\|\phi - \phi_h\|_{0,\Omega} + Ch^{k'+1}\|\phi\|_{k'+1,\Omega}. \qquad \text{(A.23)}
\end{aligned}
$$

Let us now deal with the second term of the right-hand side of (A.19). Using the H^1-stability property of the operator $\mathbb{I}_{h_\perp}$ (Section 3.1.3), we obtain

$$
\begin{aligned}
\tilde{a}\left(\phi^n_\perp - \phi^n_{h\perp}, \mathbb{I}_{h_\perp}\varphi_{f^n_\perp}\right) &= \tilde{L}^n_\perp(\mathbb{I}_{h_\perp}\varphi_{f^n_\perp}) - \tilde{L}^n_{h\perp}(\mathbb{I}_{h_\perp}\varphi_{f^n_\perp}) \\
&\le \|\tilde{\rho}^n_\perp - \tilde{\rho}^n_{h\perp}\|_{-1,\Omega_\perp}\|\mathbb{I}_{h_\perp}\varphi_{f^n_\perp}\|_{1,\Omega_\perp} \le C\|\tilde{\rho}^n_\perp - \tilde{\rho}^n_{h\perp}\|_{-1,\Omega_\perp}\|\varphi_{f^n_\perp}\|_{1,\Omega_\perp} \\
&\le C\|\tilde{\rho}^n_\perp - \tilde{\rho}^n_{h\perp}\|_{-1,\Omega_\perp}\|f^n_\perp\|_{0,\Omega_\perp} \\
&\le C\left\|\int_{\Omega_\parallel}(f^n_\parallel\tilde{\rho} - f^n_{\parallel h}\tilde{\rho}_h)\,\mathrm{d}x_\parallel\right\|_{-1,\Omega_\perp} \\
&\le C\left\|\int_{\Omega_\parallel}\tilde{\rho}(f^n_\parallel - f^n_{\parallel h})\,\mathrm{d}x_\parallel\right\|_{-1,\Omega_\perp} + C\left\|\int_{\Omega_\parallel} f^n_{\parallel h}(\tilde{\rho} - \Pi_h\tilde{\rho})\,\mathrm{d}x_\parallel\right\|_{-1,\Omega_\perp} \\
&\quad + C\left\|\int_{\Omega_\parallel} f^n_{\parallel h}(\Pi_h\tilde{\rho} - \tilde{\rho}_h)\,\mathrm{d}x_\parallel\right\|_{-1,\Omega_\perp}. \qquad \text{(A.24)}
\end{aligned}
$$

The first term of the right-hand side of (A.24) is bounded as

$$\left\| \int_{\Omega_\parallel} \tilde{\rho}(f_\parallel^n - f_{\parallel h}^n)\, \mathrm{d}x_\parallel \right\|_{-1,\Omega_\perp} \leq \|f_\parallel^n - f_{\parallel h}^n\|_{0,\Omega_\parallel} \|\tilde{\rho}\|_{0,\Omega} \leq C\|f_\parallel^n\|_{0,\Omega_\parallel} \|\tilde{\rho}\|_{0,\Omega} \leq C\|\tilde{\rho}\|_{0,\Omega}. \tag{A.25}$$

The second term of the right-hand side of (A.24) is bounded as

$$\left\| \int_{\Omega_\parallel} f_{\parallel h}^n(\tilde{\rho} - \Pi_h\tilde{\rho})\, \mathrm{d}x_\parallel \right\|_{-1,\Omega_\perp} \leq C\|f_\parallel^n\|_{0,\Omega_\parallel} \|\tilde{\rho} - \Pi_h\tilde{\rho}\|_{0,\Omega} \leq C\|\tilde{\rho} - \Pi_h\tilde{\rho}\|_{0,\Omega}. \tag{A.26}$$

The third term of the right-hand side of (A.24) is by definition

$$\left\| \int_{\Omega_\parallel} f_{\parallel h}^n(\Pi_h\tilde{\rho} - \tilde{\rho}_h)\, \mathrm{d}x_\parallel \right\|_{-1,\Omega_\perp} = \sup_{\varphi_\perp \in H_0^1(\Omega_\perp)} \frac{\left\langle \int_{\Omega_\parallel} f_{\parallel h}^n(\Pi_h\tilde{\rho} - \tilde{\rho}_h)\, \mathrm{d}x_\parallel, \varphi_\perp \right\rangle_{\Omega_\perp}}{\|\varphi_\perp\|_{1,\Omega_\perp}}.$$

Let us set $w_h = \Pi_h\tilde{\rho} - \tilde{\rho}_h$ and define W_h such that $w_h = \nabla_\perp \cdot W_h$. Then, the vector W_h can be decomposed as $W_h = \mathcal{K}_h + \mathcal{H}_h$, where the components of $\mathcal{K}_h$ are constants, while those of $\mathcal{H}_h$ are homogeneous in the variable $x_\perp$. Using this decomposition, several integration by parts and a scaling argument, we obtain

$$\begin{aligned}
\left\langle \int_{\Omega_\parallel} f_{\parallel h}^n(\Pi_h\tilde{\rho} - \tilde{\rho}_h)\, \mathrm{d}x_\parallel, \varphi_\perp \right\rangle_{\Omega_\perp} &= \int_\Omega (\Pi_h\tilde{\rho} - \tilde{\rho}_h)\varphi_\perp \otimes f_{\parallel h}^n\, \mathrm{d}x = \int_\Omega w_h\varphi_\perp \otimes f_{\parallel h}^n\, \mathrm{d}x \\
&= \int_\Omega W_h \cdot \nabla_\perp\varphi_\perp \otimes f_{\parallel h}^n\, \mathrm{d}x \\
&= \int_\Omega \mathcal{K}_h \cdot \nabla_\perp\varphi_\perp \otimes f_{\parallel h}^n\, \mathrm{d}x + \int_\Omega \mathcal{H}_h \cdot \nabla_\perp\varphi_\perp \otimes f_{\parallel h}^n\, \mathrm{d}x \\
&= \int_\Omega \mathcal{H}_h \cdot \nabla_\perp\varphi_\perp \otimes f_{\parallel h}^n\, \mathrm{d}x \\
&\leq C\|\mathcal{H}_h\|_{0,\Omega} \|\nabla_\perp\varphi_\perp\|_{0,\Omega_\perp} \|f_{\parallel h}^n\|_{0,\Omega_\parallel} \\
&\leq Ch\|w_h\|_{0,\Omega} \|\varphi_\perp\|_{1,\Omega_\perp} \\
&\leq Ch\left(\|\tilde{\rho} - \Pi_h\tilde{\rho}\|_{0,\Omega} + \|\tilde{\rho} - \tilde{\rho}_h\|_{0,\Omega}\right) \|\varphi_\perp\|_{1,\Omega_\perp}.
\end{aligned} \tag{A.27}$$

Gathering (A.25)–(A.27) we obtain

$$\left\| \int_{\Omega_\parallel} (f_\parallel^n\tilde{\rho} - f_{\parallel h}^n\tilde{\rho}_h)\, \mathrm{d}x_\parallel \right\|_{-1,\Omega_\perp} \leq C\left(\|\tilde{\rho}\|_{0,\Omega} + \|\tilde{\rho} - \Pi_h\tilde{\rho}\|_{0,\Omega} + h\|\tilde{\rho} - \tilde{\rho}_h\|_{0,\Omega}\right). \tag{A.28}$$

Now, we take $\chi_h \in V_h^k$, such that $\chi_h = \chi_{h\perp} \otimes \chi_{h\parallel}$ and observe that

$$\chi_{h\perp} \int_{\Omega_\parallel} (f_\parallel^n - f_{\parallel h}^n)\chi_{h\parallel}\, \mathrm{d}x_\parallel = 0,$$

because $f^n_{\parallel h} = \mathbb{P}_{h_\parallel} f^n_\parallel$. Therefore, using the previous equality and approximation properties in the space V^k_h, estimate (A.28) leads to

$$\begin{aligned}\left\| \int_{\Omega_\parallel} (f^n_\parallel \tilde{\rho} - f^n_{\parallel h} \tilde{\rho}_h)\, \mathrm{d}x_\parallel \right\|_{-1,\Omega_\perp} &= \left\| \int_{\Omega_\parallel} (f^n_\parallel (\tilde{\rho} - \chi_h) - f^n_{\parallel h} (\tilde{\rho}_h - \chi_h))\, \mathrm{d}x_\parallel \right\|_{-1,\Omega_\perp} \\ &\leq C \left(\|\tilde{\rho} - \chi_h\|_{0,\Omega} + \|\tilde{\rho} - \Pi_h \tilde{\rho}\|_{0,\Omega} + h \|\tilde{\rho} - \tilde{\rho}_h\|_{0,\Omega} \right) \\ &\leq C \left(h^{k+1} \left(\sum_{\beta,\sigma} \mathcal{A}_\sigma \left\| v^\beta_\sigma \right\|^2_{0,\Omega} \right)^{1/2} + h \|\tilde{\rho} - \tilde{\rho}_h\|_{0,\Omega} \right). \end{aligned} \tag{A.29}$$

Finally, gathering (A.11), (A.14), (A.17)–(A.19), (A.23)–(A.24) and (A.29), we find

$$\begin{aligned}\|\phi - \phi_h\|_{0,\Omega} \leq C \Bigg\{ &(\varepsilon + \varepsilon_h) \|\phi - \phi_h\|_{0,\Omega} + h \|\phi - \phi_h\|_{L^2\left(\Omega_\parallel; H^1_0(\Omega_\perp)\right)} + h^{k'+1} \|\phi\|_{k'+1,\Omega} \\ &+ h^{k+1} \left(\sum_{\beta,\sigma} \mathcal{A}_\sigma \left\| v^\beta_\sigma \right\|^2_{0,\Omega} \right)^{1/2} + h \left(\sum_{\beta,\sigma} \mathcal{A}_\sigma \left\| v^\beta_\sigma - v^\beta_{\sigma h} \right\|^2_{0,\Omega} \right)^{1/2} \Bigg\}. \end{aligned}$$

If $(\varepsilon + \varepsilon_h)$ is small enough, the associated term could be absorbed into the left-hand side of the previous inequality. Therefore (4.1) with $|\alpha| = 1$ leads to (4.1) with $|\alpha| = 0$.

A.2 *Proof of Proposition 4.4*

In this section, we show error estimates in the L^∞-norm for the approximation of the electrical potential, which is required by our analysis. For this purpose we recall a result concerning the L^p-regularity estimates for elliptic partial differential equations of second order that are needed for the proof and were established through several articles (Agmon *et al.*, 1959, 1964; Schechter, 1963; Campanato & Stampacchia, 1965; Dauge, 1988; Gilbarg & Trudinger, 1998). Let us introduce the following second-order elliptic Neumann problem. We assume that $a_0,\ b_0 \in W^{1,\infty}(\Omega_\perp)$, with $a_0 \geq a^0_0 > 0$ and $b_0 \geq b^0_0 > 0$. Let $g \in L^p(\Omega_\perp)$ with $1 < p < \infty$. Then the Neumann problem

$$\begin{cases} \mathcal{L}\varphi = g \ \text{ in } \ \Omega_\perp, \\ \dfrac{\partial \varphi}{\partial \nu_\perp} = 0 \ \text{ on } \ \partial\Omega_\perp, \end{cases} \tag{A.30}$$

where

$$\mathcal{L}\psi = -\nabla_\perp \cdot (a_0 \nabla_\perp \psi) + b_0 \psi \quad \forall \psi \in W^{2,p}(\Omega_\perp), \tag{A.31}$$

has a unique solution $\varphi \in W^{2,p}(\Omega_\perp)$ which satisfies the estimate

$$\|\varphi\|_{2,p,\Omega_\perp} \leq C(p) \|g\|_{0,p,\Omega_\perp}.$$

If we trace the constant in the proof (e.g., see Gilbarg & Trudinger, 1998), we observe that $C(p) = c(p-1)$ if $2 \leq p < \infty$ and $C(p) = c/(p-1)$ if $1 < p \leq 2$. This dependence on p is the same as for the constant arising from the Marcinkiewicz interpolation theorem (Gilbarg & Trudinger (1998), Theorem 9.8) and the Calderón–Zygmund theory of singular integral operators (Calderón & Zygmund, 1952; Stein, 1970), which are the main tools for proving this *a priori* estimate.

If now $g = \partial_\perp^\alpha f$ with $|\alpha| = 1$ and $f \in L^p(\Omega_\perp)$ then the Neumann problem (A.30) has a unique solution $\varphi \in W^{1,p}(\Omega_\perp)$, which satisfies the estimate

$$\|\varphi\|_{1,p,\Omega_\perp} \leq C(p)\|f\|_{0,p,\Omega_\perp}. \tag{A.32}$$

Therefore, the above results lead to the following statement. If $g \in W^{-1,p}(\Omega_\perp)$ then the Neumann problem (A.30) has a unique solution $\varphi \in W^{1,p}(\Omega_\perp)$ which satisfies the estimate

$$\|\varphi\|_{1,p,\Omega_\perp} \leq C(p)\|g\|_{-1,p,\Omega_\perp}. \tag{A.33}$$

Proof. Let $u \in W_0^{1,p}(\Omega_\perp)/\{\text{constant function}\}$ be the unique solution of the following variational problem: for all $v \in W_0^{1,p}(\Omega_\perp)$, $\langle \nabla_\perp u, \nabla_\perp v\rangle_{\Omega_\perp} = \langle g, v\rangle_{\Omega_\perp}$, with $g \in W^{-1,p}(\Omega_\perp)$ and $\langle g, 1\rangle_{\Omega_\perp} = 0$. In other words $u \in W_0^{1,p}(\Omega_\perp)$ is the weak solution to the following Dirichet problem: $-\Delta u = g$ on $\Omega_\perp$, with $u = 0$ on $\partial\Omega_\perp$ and $g \in W^{-1,p}(\Omega_\perp)$. On the one hand, using the Poincaré inequality, we get $\|u\|_{1,p,\Omega_\perp} \leq \kappa|u|_{1,p,\Omega_\perp}$. On the other hand, for any vector-valued function $V \in L^{p^*}(\Omega_\perp)$ (with $1/p + 1/p^* = 1$) one can find a scalar-valued function $v \in W_0^{1,p^*}(\Omega_\perp)$ such that $V = \nabla_\perp v$ and one has

$$\begin{aligned}|u|_{1,p,\Omega_\perp} = \|\nabla_\perp u\|_{0,p,\Omega_\perp} &\leq \sup_{V\in L^{p^*}(\Omega_\perp)} \frac{\langle \nabla_\perp u, V\rangle_{\Omega_\perp}}{\|V\|_{0,p^*,\Omega_\perp}} \leq \sup_{v\in W_0^{1,p^*}(\Omega_\perp)} \frac{|\langle \nabla_\perp u, \nabla_\perp v\rangle_{\Omega_\perp}|}{\|\nabla_\perp v\|_{0,p^*,\Omega_\perp}}\\ &\leq \kappa \sup_{v\in W_0^{1,p^*}(\Omega_\perp)} \frac{|\langle g, v\rangle_{\Omega_\perp}|}{\|v\|_{1,p^*,\Omega_\perp}} \leq \kappa\|g\|_{-1,p,\Omega_\perp}.\end{aligned}$$

Therefore, we get $\|u\|_{1,p,\Omega_\perp} \leq \kappa^2\|g\|_{-1,p,\Omega_\perp}$. Let us now write $g = \nabla_\perp \cdot f \in W^{-1,p}(\Omega_\perp)$, where $f \in L^p(\Omega_\perp)$ and $f = -\nabla_\perp u$ with $u \in W_0^{1,p}(\Omega_\perp)$. Then from (A.32) and the previous development, we obtain

$$\|\varphi\|_{1,p,\Omega_\perp} \leq C(p)\|f\|_{0,p,\Omega_\perp} \leq C(p)\|\nabla_\perp u\|_{0,p,\Omega_\perp} \leq C(p)\|u\|_{1,p,\Omega_\perp} \leq C(p)\|g\|_{-1,p,\Omega_\perp},$$

which proves (A.33). □

A.2.1 $L^\infty\left(\Omega_\parallel;\, W^{1,\infty}(\Omega_\perp)\right)$*-error estimate for* ϕ_h. In order to show (4.2) with $|\alpha| = 1$, it is sufficient to show that, for $|\alpha| = 1$,

$$\begin{aligned}\|\partial_\perp^\alpha(\phi(t) - \phi_h(t))\|_{0,\infty,\Omega} \leq C_\infty\|\phi(t)\|_{L^\infty(\Omega_\parallel;\,W^{1,\infty}(\Omega_\perp))} + C_\infty h^{\min(k,k')+1/2}|\ln h|\\ + C_\infty h^{-1/2}|\ln h|\left(\sum_{\beta,\sigma}\mathcal{A}_\sigma\left\|v_\sigma^\beta(t) - v_{\sigma h}^\beta(t)\right\|_{0,\Omega}^2\right)^{1/2}.\end{aligned} \tag{A.34}$$

Actually, for any $\chi_h \in \Phi_h^{k'}$, if we consider $\phi - \chi_h$ instead of ϕ, and thus we rewrite $\phi - \phi_h = (\phi - \chi_h) - (\phi_h - \chi_h)$, then we obtain (4.2) with $|\alpha| = 1$. Indeed using approximation properties of the space $\Phi_h^{k'}$ (Section 3.1.3), we find

$$\inf_{\chi_h\in\Phi_h^{k'}}\|\phi - \chi_h\|_{0,\infty,\Omega} \leq Ch^{k'+1}\|\phi\|_{k'+1,\infty,\Omega}.$$

As in Schatz & Wahlbin (1995, Appendix (A.5)), for any $x_0 = (x_{0\perp}, x_{0\parallel}) \in K$, with $K \in \mathcal{M}_h$, we can construct a function $\tilde{\delta}_{\alpha,x_0} \in \mathscr{C}^1(K)$, such that

$$\left\{\begin{array}{l} \tilde{\delta}_{\alpha,x_0}(x) = \tilde{\delta}_{\perp\alpha,x_{0\perp}}(x_\perp) \otimes \tilde{\delta}_{\parallel\alpha,x_{0\parallel}}(x_\parallel), \quad \mathrm{supp}(\tilde{\delta}_{\alpha,x_0}) \subset \overline{K}\,, \quad \tilde{\delta}_{\parallel\alpha,x_{0\parallel}} \in \mathbb{P}^1_m(K_\parallel),\ m \le k', \\ \forall \chi_h \in \Phi_h^{k'},\ \partial_\perp^\alpha \chi_h(x_0) = \displaystyle\int_K \partial_\perp^\alpha \chi_h(x) \tilde{\delta}_{\alpha,x_0}(x)\,\mathrm{d}x, \ \text{ for } |\alpha| = 1, \\ h^{-1} \left\| \tilde{\delta}_{\tau\alpha,x_{0\tau}} \right\|_{-1,q,K_\tau} + \left\| \tilde{\delta}_{\tau\alpha,x_{0\tau}} \right\|_{0,q,K_\tau} + h \left| \tilde{\delta}_{\tau\alpha,x_{0\tau}} \right|_{1,q,K_\tau} \le C h^{-d_\tau(1-1/q)}, \\ \text{with } \ \tau \in \{\perp, \parallel\},\ d_\perp = 2 \ \text{ and } \ d_\parallel = 1. \end{array}\right. \tag{A.35}$$

If we set $x_0 = (x_{0\perp}, x_{0\parallel}) \in K$, with $K \in \mathcal{M}_h$ and choose $\tilde{\delta}_{\alpha,x_0}$ such as in (A.35) then for $|\alpha| = 1$, we obtain

$$\partial_\perp^\alpha \phi_h(x_0) = \langle \partial_\perp^\alpha \phi_h, \tilde{\delta}_{\alpha,x_0} \rangle_\Omega = \langle \partial_\perp^\alpha \phi, \tilde{\delta}_{\alpha,x_0} \rangle_\Omega + \langle \phi - \phi_h, \partial_\perp^\alpha \tilde{\delta}_{\alpha,x_0} \rangle_\Omega. \tag{A.36}$$

By (A.35), the first term of the right-hand side of (A.36) can be bounded as

$$\langle \partial_\perp^\alpha \phi, \tilde{\delta}_{\alpha,x_0} \rangle_\Omega \le \|\phi\|_{L^\infty(\Omega_\parallel;\, W^{1,\infty}(\Omega_\perp))} \left\| \tilde{\delta}_{\alpha,x_0} \right\|_{0,1,K} \le C \|\phi\|_{L^\infty(\Omega_\parallel;\, W^{1,\infty}(\Omega_\perp))}. \tag{A.37}$$

To estimate the second term of the right-hand side of (A.36), we introduce the following Neumann problem. Let v be determined as the unique solution of the problem

$$\left\{\begin{array}{l} \mathcal{L}v = \partial_\perp^\alpha \tilde{\delta}_{\perp\alpha,x_{0\perp}} \ \text{ in } \ \Omega_\perp, \\ \dfrac{\partial v}{\partial \nu_\perp} = 0 \ \text{ on } \ \partial\Omega_\perp, \end{array}\right. \tag{A.38}$$

where the self-adjoint linear differential operator $\mathcal{L}$ is defined by (A.31). Let $v_h \in \bar{\Phi}_{h_\perp}^{k'}$ be the unique solution of the variational problem

$$\tilde{a}(\psi_h, v - v_h) = 0 \quad \forall \psi_h \in \bar{\Phi}_{h_\perp}^{k'}, \tag{A.39}$$

where the bilinear form $\tilde{a}(\cdot,\cdot)$ is defined by (A.16). Then, we obtain

$$\begin{aligned} \left| \left\langle \phi - \phi_h, \partial_\perp^\alpha \tilde{\delta}_{\alpha,x_0} \right\rangle_\Omega \right| &\le \left| \left\langle (\phi - \phi_h) \tilde{\delta}_{\parallel\alpha,x_{0\parallel}}, \partial_\perp^\alpha \tilde{\delta}_{\perp\alpha,x_{0\perp}} \right\rangle_\Omega \right| \\ &\le \left| \left\langle (\phi - \phi_h) \tilde{\delta}_{\parallel\alpha,x_{0\parallel}}, \mathcal{L}v \right\rangle_\Omega \right| \\ &\le \left| a\left((\phi - \phi_h) \tilde{\delta}_{\parallel\alpha,x_{0\parallel}}, v \right) \right| \\ &\le \left| a\left(\phi - \phi_h, v \otimes \tilde{\delta}_{\parallel\alpha,x_{0\parallel}} \right) \right| \\ &\le \left| a\left(\phi - \phi_h, (v - v_h) \otimes \tilde{\delta}_{\parallel\alpha,x_{0\parallel}} \right) \right| + \left| a\left(\phi - \phi_h, v_h \otimes \tilde{\delta}_{\parallel\alpha,x_{0\parallel}} \right) \right| \\ &\le \left| a\left(\phi - \phi_h, (v - v_h) \otimes \tilde{\delta}_{\parallel\alpha,x_{0\parallel}} \right) \right| + \left| (L - L_h) \left(v_h \otimes \tilde{\delta}_{\parallel\alpha,x_{0\parallel}} \right) \right|. \end{aligned}$$

Now, we claim that $a\left(\phi_h, (v - v_h) \otimes \tilde{\delta}_{\|\alpha, x_{0\|}}\right) = 0$. Indeed since $\tilde{\phi}_h \in \bar{\Phi}^{k'}_{h_\perp}$ with the definition

$$\tilde{\phi}_h = \int_{\Omega_\|} \phi_h \tilde{\delta}_{\|\alpha, x_{0\|}} \, \mathrm{d}x_\|,$$

we find

$$a\left(\phi_h, (v - v_h) \otimes \tilde{\delta}_{\|\alpha, x_{0\|}}\right) = \tilde{a}\left(\tilde{\phi}_h, v - v_h\right) = 0,$$

where we have used the orthogonality relation (A.39). Therefore, we obtain

$$\begin{aligned}
\left|\left\langle \phi - \phi_h, \partial_\perp^\alpha \tilde{\delta}_{\alpha, x_0} \right\rangle_\Omega\right| &\leq \left|a\left(\phi, (v - v_h) \otimes \tilde{\delta}_{\|\alpha, x_{0\|}}\right)\right| + \left|(L - L_h)\left(v_h \otimes \tilde{\delta}_{\|\alpha, x_{0\|}}\right)\right| \\
&\leq \|\phi\|_{L^\infty(\Omega_\|; W^{1,\infty}(\Omega_\perp))} \left\|(v - v_h) \otimes \tilde{\delta}_{\|\alpha, x_{0\|}}\right\|_{L^1(\Omega_\|; W^{1,1}(\Omega_\perp))} \\
&\quad + \left|(L - L_h)\left(v_h \otimes \tilde{\delta}_{\|\alpha, x_{0\|}}\right)\right| \\
&\leq \|\phi\|_{L^\infty(\Omega_\|; W^{1,\infty}(\Omega_\perp))} \|(v - v_h)\|_{1,1,\Omega_\perp} \left\|\tilde{\delta}_{\|\alpha, x_{0\|}}\right\|_{0,1,\Omega_\|} \\
&\quad + \left|(L - L_h)\left(v_h \otimes \tilde{\delta}_{\|\alpha, x_{0\|}}\right)\right|.
\end{aligned} \tag{A.40}$$

On the one hand using (A.35) and on the other hand using Schatz & Wahlbin (1995, Lemma 3.1), we obtain, respectively,

$$\left\|\tilde{\delta}_{\|\alpha, x_{0\|}}\right\|_{0,1,\Omega_\|} \leq C < \infty \quad \text{and} \quad \|v - v_h\|_{1,1,\Omega_\perp} \leq C < \infty. \tag{A.41}$$

By (A.40)–(A.41), we infer

$$\left|\left\langle \phi - \phi_h, \partial_\perp^\alpha \tilde{\delta}_{\alpha, x_0} \right\rangle_\Omega\right| \leq C\|\phi\|_{L^\infty(\Omega_\|; W^{1,\infty}(\Omega_\perp))} + \left|(L - L_h)\left(v_h \otimes \tilde{\delta}_{\|\alpha, x_{0\|}}\right)\right|. \tag{A.42}$$

Let us now estimate the second term of the right-hand side of (A.42). With p, $p^* \in \mathbb{N}^*$ and $1/p + 1/p^* = 1$, we obtain

$$\begin{aligned}
\left|(L - L_h)\left(v_h \otimes \tilde{\delta}_{\|\alpha, x_{0\|}}\right)\right| &\leq \left|\left\langle \tilde{\rho} - \tilde{\rho}_h, v_h \otimes \tilde{\delta}_{\|\alpha, x_{0\|}} \right\rangle_\Omega\right| \\
&\leq \|\tilde{\rho} - \tilde{\rho}_h\|_{0,p,\Omega} \left\|\tilde{\delta}_{\|\alpha, x_{0\|}}\right\|_{0,p^*,\Omega_\|} \|v_h\|_{0,p^*,\Omega_\perp}.
\end{aligned} \tag{A.43}$$

From the continuous embedding $W^{m,q}(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d)$, with $1/p^* = 1/q - m/d$ and $mq < d$, if we choose $m = q = 1$, $d = 2$ then we get $p = p^* = 2$. Therefore, using (A.32), (A.35) and (A.41), for

$1 < r \leq 2$, we obtain

$$\begin{aligned}
\|v_h\|_{0,p^*,\Omega_\perp} &\leq \|v_h\|_{1,1,\Omega_\perp} \\
&\leq \|v - v_h\|_{1,1,\Omega_\perp} + \|v\|_{1,1,\Omega_\perp} \\
&\leq C\left(1 + \|v\|_{1,1,\Omega_\perp}\right) \leq C\left(1 + \|v\|_{1,r,\Omega_\perp}\right) \\
&\leq C\left(1 + \left\|\tilde{\delta}_{\perp\alpha,x_{0\perp}}\right\|_{0,r,\Omega_\perp}\right) \\
&\leq C\left(1 + \frac{h^{-2(1-1/r)}}{r-1}\right) \leq C\frac{h^{-2(1-1/r)}}{r-1}.
\end{aligned} \tag{A.44}$$

By (A.35) we obtain

$$\left\|\tilde{\delta}_{\|\alpha,x_{0\|}}\right\|_{0,p^*,\Omega_\|} \leq Ch^{-1(1-1/p^*)} \leq Ch^{-1/2},$$

so that, using (A.43)–(A.44), we find

$$\left|(L - L_h)\left(v_h \otimes \tilde{\delta}_{\|\alpha,x_{0\|}}\right)\right| \leq \|\tilde{\rho} - \tilde{\rho}_h\|_{0,\Omega} \frac{h^{-2(1-1/r)-1/2}}{r-1}. \tag{A.45}$$

By choosing $r = 1 + 1/\ln(1/h)$ and noting that for h small enough,

$$h^{-2(1-1/r)} = h^{-2\left(1-\frac{1}{1+1/\ln(1/h)}\right)} \leq Ch^{-2/|\ln h|} \leq C < \infty,$$

estimate (A.45) becomes

$$\left|(L - L_h)\left(v_h \otimes \tilde{\delta}_{\|\alpha,x_{0\|}}\right)\right| \leq Ch^{-1/2}|\ln h|\,\|\tilde{\rho} - \tilde{\rho}_h\|_{0,\Omega},$$

from which, using (A.11), (A.36)–(A.37) and (A.42), we obtain the desired result (A.34).

A.2.2 *$L^\infty(\Omega)$-error estimate for ϕ_h.* In order to show (4.2) with $|\alpha| = 0$, it is sufficient to show the inequality

$$\begin{aligned}
\|\phi(t) - \phi_h(t)\|_{0,\infty,\Omega} \leq C_\infty \|\phi(t)\|_{0,\infty,\Omega} |\ln h|^{\varkappa} + C_\infty h^{\min(k,k')+3/2} |\ln h| \\
+ C_\infty h^{1/2} |\ln h| \left(\sum_{\beta,\sigma} \mathcal{A}_\sigma \left\|v^\beta_\sigma(t) - v^\beta_{\sigma h}(t)\right\|^2_{0,\Omega}\right)^{1/2}.
\end{aligned} \tag{A.46}$$

Actually, for any $\chi_h \in \Phi^{k'}_h$, if we consider $\phi - \chi_h$ instead of ϕ, and thus we rewrite $\phi - \phi_h = (\phi - \chi_h) - (\phi_h - \chi_h)$, then we obtain (4.2) with $|\alpha| = 0$. Indeed by using the approximation properties of the space $\Phi^{k'}_h$ (Section 3.1.3), we find

$$\inf_{\chi_h \in \Phi^{k'}_h} \|\phi - \chi_h\|_{0,\infty,\Omega} \leq Ch^{k'+1} \|\phi\|_{k'+1,\infty,\Omega}.$$

As in Schatz & Wahlbin (1995, Appendix (A.5)), for any $x_0 = (x_{0\perp}, x_{0\parallel}) \in K$, with $K \in \mathcal{M}_h$, we can construct a function $\tilde{\delta}_{x_0} \in \mathscr{C}^1(K)$ such that

$$\begin{cases} \tilde{\delta}_{x_0}(x) = \tilde{\delta}_{\perp, x_{0\perp}}(x_\perp) \otimes \tilde{\delta}_{\parallel, x_{0\parallel}}(x_\parallel), \quad \mathrm{supp}(\tilde{\delta}_{x_0}) \subset \bar{K}\,, \quad \tilde{\delta}_{\parallel, x_{0\parallel}} \in \mathbb{P}^1_m(K_\parallel),\ m \leq k', \\ \forall \chi_h \in \Phi_h^{k'},\ \ \chi_h(x_0) = \displaystyle\int_K \chi_h(x)\tilde{\delta}_{x_0}(x)\,\mathrm{d}x, \\ h^{-1}\left\|\tilde{\delta}_{\tau, x_{0\tau}}\right\|_{-1,q,K_\tau} + \left\|\tilde{\delta}_{\tau, x_{0\tau}}\right\|_{0,q,K_\tau} + h\left|\tilde{\delta}_{\tau, x_{0\tau}}\right|_{1,q,K_\tau} \leq Ch^{-d_\tau(1-1/q)}, \\ \text{with}\ \ \tau \in \{\perp, \parallel\},\ \ d_\perp = 2\ \ \text{and}\ \ d_\parallel = 1. \end{cases} \tag{A.47}$$

If we set $x_0 = (x_{0\perp}, x_{0\parallel}) \in K$, with $K \in \mathcal{M}_h$ and choose $\tilde{\delta}_{x_0}$ such that in (A.47) then we obtain

$$\phi_h(x_0) = \langle \phi_h, \tilde{\delta}_{x_0} \rangle_\Omega = \langle \phi, \tilde{\delta}_{x_0} \rangle_\Omega + \langle \phi - \phi_h, \tilde{\delta}_{x_0} \rangle_\Omega. \tag{A.48}$$

By (A.35), the first term of the right-hand side of (A.48) can be bounded as

$$\langle \phi, \tilde{\delta}_{x_0} \rangle_\Omega \leq \|\phi\|_{0,\infty,\Omega} \left\|\tilde{\delta}_{x_0}\right\|_{0,1,K} \leq C\|\phi\|_{0,\infty,\Omega}. \tag{A.49}$$

To estimate the second term of the right-hand side of (A.48), we introduce the following Neumann problem. Let v be determined as the unique solution of the problem

$$\begin{cases} \mathcal{L}v = \tilde{\delta}_{\perp, x_{0\perp}} \ \ \text{in}\ \ \Omega_\perp, \\ \dfrac{\partial v}{\partial \nu_\perp} = 0\ \ \text{on}\ \ \partial\Omega_\perp, \end{cases} \tag{A.50}$$

where the self-adjoint linear differential operator $\mathcal{L}$ is defined by (A.31). Let $v_h \in \bar{\Phi}_{h_\perp}^{k'}$ be the unique solution of the variational problem

$$\tilde{a}(\psi_h, v - v_h) = 0 \quad \forall \psi_h \in \bar{\Phi}_{h_\perp}^{k'}, \tag{A.51}$$

where the bilinear form $\tilde{a}(\cdot,\cdot)$ is defined by (A.16). Therefore, we obtain

$$\begin{aligned} \left|\left\langle \phi - \phi_h, \tilde{\delta}_{x_0} \right\rangle_\Omega\right| &\leq \left|\left\langle (\phi - \phi_h)\tilde{\delta}_{\parallel, x_{0\parallel}}, \mathcal{L}v \right\rangle_\Omega\right| \\ &\leq \left|a\left((\phi - \phi_h)\tilde{\delta}_{\parallel, x_{0\parallel}}, v\right)\right| \\ &\leq \left|a\left(\phi - \phi_h, v \otimes \tilde{\delta}_{\parallel, x_{0\parallel}}\right)\right| \\ &\leq \left|a\left(\phi - \phi_h, (v - v_h) \otimes \tilde{\delta}_{\parallel, x_{0\parallel}}\right)\right| + \left|a\left(\phi - \phi_h, v_h \otimes \tilde{\delta}_{\parallel, x_{0\parallel}}\right)\right| \\ &\leq \left|a\left(\phi - \phi_h, (v - v_h) \otimes \tilde{\delta}_{\parallel, x_{0\parallel}}\right)\right| + \left|(L - L_h)\left(v_h \otimes \tilde{\delta}_{\parallel\alpha, x_{0\parallel}}\right)\right| \\ &\leq \left|a\left(\phi - \phi_h, (v - v_h) \otimes \tilde{\delta}_{\parallel, x_{0\parallel}}\right)\right| + \left|(L - L_h)\left((v - v_h) \otimes \tilde{\delta}_{\parallel\alpha, x_{0\parallel}}\right)\right| \\ &\quad + \left|(L - L_h)\left(v \otimes \tilde{\delta}_{\parallel\alpha, x_{0\parallel}}\right)\right|. \end{aligned} \tag{A.52}$$

Let us deal with the first term of the right-hand side of (A.52). Introducing

$$\tilde{\phi} = \int_{\Omega_\parallel} \phi \tilde{\delta}_{\parallel, x_{0\parallel}} \, \mathrm{d}x_\parallel \quad \text{and} \quad \tilde{\phi}_h = \int_{\Omega_\parallel} \phi_h \tilde{\delta}_{\parallel, x_{0\parallel}} \, \mathrm{d}x_\parallel,$$

we find

$$a\left(\phi - \phi_h, (v - v_h) \otimes \tilde{\delta}_{\parallel, x_{0\parallel}}\right) = a\left(\tilde{\phi}, v - v_h\right) - a\left(\tilde{\phi}_h, v - v_h\right) = a\left(\tilde{\phi}, v - v_h\right),$$

since $a\left(\tilde{\phi}_h, v - v_h\right) = 0$, by using (A.51) with $\psi_h = \tilde{\phi}_h \in \bar{\Phi}^{k'}_{h_\perp}$. From the following trace inequality of Schatz & Wahlbin (1977, 1995, Appendix (A0)), for all $f \in W^{2,1}(K_\perp)$ with $K_\perp \in \mathcal{T}_{h_\perp}$,

$$\int_{\partial K_\perp} |\nabla_\perp f| \, \mathrm{d}s \leq C\left(\frac{1}{h}|f|_{1,1,K_\perp} + |f|_{2,1,K_\perp}\right),$$

we obtain

$$\begin{aligned}
\left|a\left(\phi - \phi_h, (v - v_h) \otimes \tilde{\delta}_{\parallel, x_{0\parallel}}\right)\right| &\leq \left|a\left(\tilde{\phi}, v - v_h\right)\right| \\
&\leq \left|\sum_{K_\perp \in \mathcal{T}_{h_\perp}} \int_{K_\perp} \tilde{\phi} \mathcal{L}(v - v_h) \, \mathrm{d}x_\perp + \int_{\partial K_\perp} a_0 \tilde{\phi} \nabla_\perp (v - v_h) \cdot \nu_{K_\perp} \, \mathrm{d}s\right| \\
&\leq C(a_0, b_0) \|\tilde{\phi}\|_{0,\infty,\Omega_\perp} \left(h^{-1} \|v - v_h\|_{1,1,\Omega_\perp} + \|v - v_h\|_{2,1,\Omega_\perp}\right).
\end{aligned}$$

Now, by Schatz & Wahlbin (1977, Lemma 5.3), we have

$$\|v - v_h\|_{1,1,\Omega_\perp} \leq Ch|\ln h|^{\varkappa} \quad \text{and} \quad \|v - v_h\|_{2,1,\Omega_\perp} \leq C|\ln h|^{\varkappa}, \tag{A.53}$$

where $\varkappa = 1$ if $k' = 1$ and $\varkappa = 0$ if $k' > 1$. Therefore, for the first term of the right-hand side of (A.52), we finally obtain

$$\begin{aligned}
\left|a\left(\phi - \phi_h, (v - v_h) \otimes \tilde{\delta}_{\parallel, x_{0\parallel}}\right)\right| &\leq C\|\tilde{\phi}\|_{0,\infty,\Omega_\perp} |\ln h|^{\varkappa} \\
&\leq C \left\|\tilde{\delta}_{\parallel, x_{0\parallel}}\right\|_{0,1,\Omega_\parallel} \|\phi\|_{0,\infty,\Omega} |\ln h|^{\varkappa} \\
&\leq C\|\phi\|_{0,\infty,\Omega} |\ln h|^{\varkappa},
\end{aligned} \tag{A.54}$$

where we have used (A.35).

For the second term of the right-hand side of (A.52), using continuous Sobolev embedding, (A.53) and (A.35), we obtain

$$\begin{aligned}
\left|(L - L_h)\left((v - v_h) \otimes \tilde{\delta}_{\parallel\alpha, x_{0\parallel}}\right)\right| &\leq \left\langle \tilde{\rho} - \tilde{\rho}_h, (v - v_h) \otimes \tilde{\delta}_{\parallel\alpha, x_{0\parallel}} \right\rangle_\Omega \\
&\leq \|\tilde{\rho} - \tilde{\rho}_h\|_{0,\Omega} \left\|\tilde{\delta}_{\parallel, x_{0\parallel}}\right\|_{0,\Omega_\parallel} \|v - v_h\|_{0,\Omega_\perp} \\
&\leq Ch^{-1/2} \|\tilde{\rho} - \tilde{\rho}_h\|_{0,\Omega} \|v - v_h\|_{1,1,\Omega_\perp} \\
&\leq Ch^{1/2} |\ln h|^{\varkappa} \|\tilde{\rho} - \tilde{\rho}_h\|_{0,\Omega}.
\end{aligned} \tag{A.55}$$

For the third term of the right-hand side of (A.52), using continuous Sobolev embedding, (A.33) and (A.35), we obtain for $1 < p \leq 2$,

$$\begin{aligned}\left|(L-L_h)\left(v\otimes\tilde{\delta}_{\|\alpha,x_{0\|}}\right)\right| &\leq \left\langle\tilde{\rho}-\tilde{\rho}_h, v\otimes\tilde{\delta}_{\|\alpha,x_{0\|}}\right\rangle_{\Omega}\\ &\leq \|\tilde{\rho}-\tilde{\rho}_h\|_{0,\Omega}\left\|\tilde{\delta}_{\|,x_{0\|}}\right\|_{0,\Omega_{\|}}\|v\|_{0,\Omega_{\perp}}\\ &\leq Ch^{-1/2}\|\tilde{\rho}-\tilde{\rho}_h\|_{0,\Omega}\|v\|_{1,1,\Omega_{\perp}}\\ &\leq Ch^{-1/2}\|\tilde{\rho}-\tilde{\rho}_h\|_{0,\Omega}\|v\|_{1,p,\Omega_{\perp}}\\ &\leq C\frac{h^{-1/2}}{p-1}\|\tilde{\rho}-\tilde{\rho}_h\|_{0,\Omega}\left\|\tilde{\delta}_{\perp,x_{0\perp}}\right\|_{-1,p,\Omega_{\perp}}\\ &\leq C\frac{h^{-2(1-1/p)+1/2}}{p-1}\|\tilde{\rho}-\tilde{\rho}_h\|_{0,\Omega}\\ &\leq Ch^{1/2}|\ln h|\|\tilde{\rho}-\tilde{\rho}_h\|_{0,\Omega},\end{aligned} \tag{A.56}$$

where we have taken $p = 1 + 1/\ln(1/h)$ in the penultimate inequality. Gathering (A.48)–(A.49), (A.52), (A.54)–(A.56) and (A.11), we obtain the desired result (A.46), which completes the proof.