



ELSEVIER

Physica D 102 (1997) 195–207

PHYSICA D

On the role of high order resonances in normal forms and in separatrix splitting

Alessandro Morbidelli^{a,*}, Antonio Giorgilli^b

^a CNRS, Observatoire de la Côte d'Azur BP 229, 06304 – Nice Cedex 4, France

^b Dipartimento di Matematica dell' Università di Milano and Gruppo Nazionale di Fisica Matematica del CNR, Via Saldini 50, 20133 Milano, Italy

Received 17 February 1996; revised 26 April 1996; accepted 9 May 1996

Communicated by V.I. Arnol'd

Abstract

We discuss the role of high order resonances in the construction of normal forms for ε -close to integrable Hamiltonian systems. By heuristic considerations based on standard estimates, we show that the remainder of normal forms is dominated by the terms corresponding to the main high order resonances, and we provide a general argument to show that the size of such leading terms is exponentially small in $1/\varepsilon$. We apply this method to the problem of estimating the splitting of separatrices in resonant perturbed systems.

1. Introduction

The search of suitable normal forms is a well-developed tool in classical perturbation theory. In recent times, several methods have been devised in order to bring these methods, first used as purely formal tools, to a quantitative and more rigorous level. KAM theory and Nekhoroshev's theory are the most advanced results in this framework. However, it is well known that estimates of the remainder of normal forms typically furnish only upper bounds, and require a lot of analytical and algebraic computations.

In this paper we identify a typical structure of normal forms, which turn out to contain some critical terms with a specific size. We show in particular that this structure is "generic", in the sense discussed below.

We study the case of near to integrable canonical systems, as described, for instance, by the Hamiltonian $H(p, q, \varepsilon) = h_0(p) + \varepsilon f(p, q)$; as usual, p, q are action-angle variables, ε is a small parameter, and the Hamiltonian is assumed to be analytic. Our aim is to transform such Hamiltonian to normal form, up to some small remainder; we point out the existence of a strict relation between the estimates of the remainder and the presence of resonances of high order in the domain where the transformation to normal form is required to be analytic.

More precisely, having fixed some domain (local in the actions), we show that one can typically give the Hamiltonian the form

$$H(p, q) = h_0 + h_{\text{norm}} + h_{\text{res}} + h_{\text{nonres}}, \quad (1)$$

where h_{norm} is in normal form with respect to some resonance module, and the remainder is split into two

* Corresponding author.

parts: h_{res} , containing only high order Fourier modes which correspond to resonances inside the considered domain, but not included in the resonance module, and h_{nonres} containing all the other terms. This will be made clear by the examples discussed in Section 2. The relevant information is that h_{res} dominates with respect to h_{nonres} . The size of h_{res} , at least for what concerns the asymptotic dependence on the relevant perturbation parameters, can be predicted by simple considerations on the order of the resonances involved and on the analyticity properties of the Hamiltonian. As a consequence, this gives an a priori estimate on the remainder of normal forms. Therefore, we conclude that the choice of the domain and, consequently, the high order resonances that are present inside that domain, tell us which terms are expected to be the leading ones in the remainder.

A relevant question is whether or not these terms do actually appear, with the predicted size. Simple analytical considerations show that, in case this is not true, a small change in the original Hamiltonian will actually produce such terms. Thus, the normal form above has to be considered, in some sense, generically optimal. This will be discussed in Section 3.

In Section 4, we shall illustrate a possible application of normal forms to estimate the size of chaotic regions in case the Hamiltonian h_{norm} has homoclinic or heteroclinic connections between hyperbolic manifolds (for instance, the rapidly forced pendulum). The key point is that the size of the chaotic region close to the stable and unstable manifolds is of the same magnitude as h_{res} . We emphasize that this result is generically optimal, in the sense expressed above. Moreover, it is in agreement with the rigorous estimates given in particular cases by Poincaré–Melnikov methods.

In Section 5, we discuss possible applications to some problems which are open in the framework of classical Poincaré–Melnikov’s theory.

2. On the remainder of normal forms

We start this section by illustrating the process of reduction of the Hamiltonian to normal form (1) in a general case. Next, we shall apply our procedure

to five model examples, that can be considered as typical.

2.1. Reduction to normal form

We consider the Hamiltonian

$$H(p, q, \varepsilon) = h(p) + \varepsilon f(p, q)$$

with $q \in \mathbb{T}^n$ and $p \in \mathcal{G} \subset \mathbb{R}^n$, where \mathcal{G} is an open set. As usual, the Hamiltonian will be supposed to be real analytic, and to admit an holomorphic extension to a complex domain $\mathcal{G}_\varrho \times \mathbb{T}_\sigma^n$, where ϱ, σ are positive parameters, and

$$\mathcal{G}_\varrho = \bigcup_{p \in \mathcal{G}} B_\varrho(p),$$

$$\mathbb{T}_\sigma^n = \{q \in \mathbb{C}^n : |\text{Im } q| < \sigma\},$$

$B_\varrho(p)$ being the complex ball of center p and radius ϱ .

We basically work with the geometrical and analytical apparatus of Nekhoroshev’s theorem; see for instance [1–4]. We recall the necessary definitions. A resonance K -module \mathcal{M} is defined as a submodule of \mathbb{Z}^n satisfying (i) $\text{span}(\mathcal{M}) \cap \mathbb{Z}^n = \mathcal{M}$, and (ii) there are $s = \dim \mathcal{M}$ independent vectors k_1, \dots, k_s in \mathcal{M} satisfying $|k_j| \leq K$. The domain \mathcal{G} is assumed to be a nonresonance domain of type $(\mathcal{M}, \alpha, \varrho, K)$, i.e., for all $p \in \mathcal{G}_\varrho$ one has $|k \cdot \omega(p)| \geq \alpha$ for all $k \in \mathbb{Z}^n \setminus \mathcal{M}$ with $|k| < K$. Here, α is a positive parameter, and $\omega(p) = \partial h_0 / \partial p$ are the unperturbed frequencies of the system. We emphasize that determining the nonresonance domain, i.e., all parameters \mathcal{M} , K and α is the crucial point of the whole procedure.

The first step is the determination of the effective size of the perturbation. Here we sketch the general scheme, but the procedure will be better illustrated for the specific examples below. We consider two different cases, namely the nonresonant case $\mathcal{M} = \{0\}$ and the resonant one.

If \mathcal{M} is nontrivial, we give the Hamiltonian the form

$$H(p, q) = h_0(p) + h_1(p, q) + h_2(p, q), \quad (2)$$

where: (i) $h_0(p)$ represents an integrable system, which does not necessarily coincide with $h(p)$ above;

(ii) $h_1(p, q)$ contains only resonant modes in \mathcal{M} and has zero average (the average part being included in h_0); (iii) $h_2(p, q)$ is a remainder, the size of which must not exceed the size of h_1 . To this end, we put the Hamiltonian in Birkhoff's normal form (i.e., remove all dependencies on the angles) until condition (iii) is satisfied. Then, we define ε_{eff} as the size of the field generated by h_1 .

In the case $\mathcal{M} = \{0\}$, with a sequence of transformations to Birkhoff's normal form, we give the Hamiltonian the form

$$H(p, q) = h_0(p) + h_2(p, q) \quad (3)$$

in such a way that the leading term in h_2 is a resonant Fourier mode, i.e., $|k \cdot \omega(p)| < \alpha$ at some point $p \in \mathcal{G}$; by the nonresonance hypothesis on the domain, we evidently have $|k| \geq K$. Then we define ε_{eff} as the size of the field generated by h_2 .

The second step consists in putting the Hamiltonian in the best normal form with respect to the resonance module \mathcal{M} , trying to minimize each term in h_2 . Precisely, we look for a form

$$H(p, q) = h_0(p) + h_{\text{norm}}(p, q) + R(p, q),$$

where h_{norm} is in normal form with respect to the resonance module \mathcal{M} , i.e., it contains only Fourier modes $k \in \mathcal{M}$, and the remainder $R(p, q)$ is as small as possible. This is the usual normalization procedure. However, we look more closely at the remainder $R(p, q)$, splitting it as $R = h_{\text{res}} + h_{\text{nonres}}$, where h_{res} contains Fourier modes k which are resonant, namely $|k \cdot \omega(p)| < \alpha$ for some $p \in \mathcal{G}$, but with $|k| > K$, while h_{nonres} contains all nonresonant Fourier modes. In order to clarify this point, we show how to perform the first normalization step. We determine the generating function χ by solving the equation

$$\omega(p) \cdot \frac{\partial \chi}{\partial p} + h_{\text{norm}} + h_{\text{res}} = h_2,$$

according to the following algorithm: let k be a Fourier mode of h_2 ; if $|k \cdot \omega(p)| > \alpha$ for all $p \in \mathcal{G}$, then put the mode in the generating function after dividing it by $ik \cdot \omega(p)$; else put the mode in h_{norm} if $k \in \mathcal{M}$, or in h_{res} if $k \notin \mathcal{M}$. Recalling that \mathcal{G} is a nonresonance domain of type $(\mathcal{M}, \alpha, \varrho, K)$, one deduces that h_{res} contains only

modes $|k| > K$. By performing the transformation we reduce the Hamiltonian to the form (1) with a remainder h_{nonres} . This step can be formally iterated an arbitrary number of times, but quantitative estimates show that we must stop at some point. Indeed, with standard estimates one finds that after r steps the size of the remainder is $O(r! \varepsilon_{\text{eff}}^r / \alpha^r)$; choosing an optimal order $r \sim \alpha / \varepsilon_{\text{eff}}$ one reduces the size of h_{nonres} to $O(\exp(-(\alpha / \varepsilon_{\text{eff}})))$.

We now proceed by further reducing the size of h_{nonres} . To this end, we essentially repeat the procedure of the second step above for increasing values of α ; more precisely, we try to remove from h_{nonres} all modes with $|k \cdot \omega(p)| > \alpha' > \alpha$. According to the general estimates above, the optimal order r increases with α' , which makes our procedure consistent. It is evident that this will affect only h_{nonres} , giving it the form of a Fourier expansion with coefficients decaying exponentially with α' . That is, the weight of a Fourier mode in h_{nonres} decreases as $\exp(-|k \cdot \omega| / \varepsilon_{\text{eff}})$.

We turn now to estimating h_{res} . It is enough to point out two elementary properties. Firstly, the transformation to normal form can be proven to be convergent in a domain, say, $\mathcal{G}_{\varrho/2} \times \mathbb{T}_{\sigma/2}^n$, where the transformed Hamiltonian turns out to be holomorphic. Secondly, the supremum norm of the Hamiltonian is not changed by the transformation. Using the well-known property of exponential decay of coefficients in the Fourier expansion of an analytic function we immediately conclude that the resonant Fourier mode k in h_{res} has a coefficient bounded by $O(\exp(-\frac{1}{2}|k|\sigma))$.

The problem now is how to compare the size of h_{res} with that of h_{nonres} . In particular, the interesting case is when h_{res} dominates h_{nonres} . With a proper choice of α this can be achieved, as will be illustrated in the examples below.

2.2. The periodically forced pendulum

We consider the Hamiltonian

$$H(p, q) = \frac{1}{2}p^2 - \varepsilon(1 + \beta \cos t) \cos q \quad (4)$$

with $p \in \mathbb{R}$, $q \in \mathbb{T}$, ε a small parameter and β a real parameter, not necessarily small. With a standard procedure, we extend the phase space introducing

canonically conjugated variables Λ, λ , with $\lambda = t$, and write the Hamiltonian as

$$H = \Lambda + \frac{1}{2}p^2 - \varepsilon(1 + \beta \cos \lambda) \cos q, \quad (5)$$

where $\Lambda \in \mathbb{R}$. It is interesting to study the local domain in the neighbourhood of $p = 0$, where q is a slow angle. Thus, we look for a normal form which does not depend on the fast angle λ . Formally, ordering the angle variables as (q, λ) the resonance module \mathcal{M} is generated by the basis $(1, 0)$.

The first step of Section 2.1 is performed as follows. If $\beta < 1$ we just put $h_0 = \Lambda + \frac{1}{2}p^2$, $h_1 = -\varepsilon \cos q$ and $h_2 = -\varepsilon\beta \cos \lambda \cos q$, so that $\varepsilon_{\text{eff}} = \varepsilon$. If $1 < \beta < 1/\varepsilon$, write

$$H = \Lambda + \frac{1}{2}p^2 - \frac{1}{2}\varepsilon\beta[\cos(\lambda + q) + \cos(\lambda - q)] - \varepsilon \cos q$$

and transform it with the generating function

$$\chi_1 = \frac{\varepsilon\beta}{2} \left[\frac{\sin(\lambda + q)}{1 + p} + \frac{\sin(\lambda - q)}{1 - p} \right].$$

This gives the transformed Hamiltonian $\exp(L_{\chi_1})H$ the form (2) with

$$h_0 = \Lambda + \frac{p^2}{2} + \frac{\varepsilon^2\beta^2(1 + p^2)}{8(1 - p^2)^2},$$

$$h_1 = -\frac{\varepsilon^2\beta^2(1 + p^2)}{8(1 - p^2)^2} \cos(2q) - \varepsilon \cos q$$

and with h_2 containing nonresonant leading terms with coefficients $\varepsilon^2\beta^2$. Now, if $\beta < 1/\sqrt{\varepsilon}$ then the dominant coefficient in h_1 is ε , so that $\varepsilon_{\text{eff}} = \varepsilon$; if instead $\beta > 1/\sqrt{\varepsilon}$ then $\varepsilon_{\text{eff}} = \varepsilon^2\beta^2$.

We now determine the domain and the parameter K . The natural choice for the domain \mathcal{G} is $|p| < O(\sqrt{\varepsilon_{\text{eff}}})$, namely the region containing the separatrices of the unperturbed pendulum $h_0 + h_1$. We also choose K as the lowest order of a mode which is resonant in \mathcal{G} but does not belong to the module \mathcal{M} . Considering the resonance relation $|2l\sqrt{\varepsilon_{\text{eff}}} - m| < \alpha$ for some positive α (that can be chosen independent of ε), we see that the lowest order resonance is given by $m = 1$ and $l \sim (1 - \alpha)/(2\sqrt{\varepsilon_{\text{eff}}})$. Thus, we have $K \sim 1/\sqrt{\varepsilon_{\text{eff}}}$, provided $\alpha < 1$. A straightforward application of the argument of Section 2.1 gives $h_{\text{res}} = O(\exp(-1/\sqrt{\varepsilon_{\text{eff}}}))$ and

$h_{\text{nonres}} = O(\exp(-\alpha/\varepsilon_{\text{eff}}))$, so that h_{res} is expected to dominate with respect to h_{nonres} . Thus the estimate of the remainder of the normal form is dominated by

$$h_{\text{res}} = O(\exp(-1/\sqrt{\varepsilon_{\text{eff}}}))$$

We point out that the well-known model of the rapidly forced pendulum is actually reduced to a model similar to the perturbed rotator above, via a canonical transformation. Indeed, it is known that the usually studied equation

$$\ddot{q} + \sin q = \beta \sin(t/\varepsilon) \quad (6)$$

can be derived from the Hamiltonian

$$H(p, \Lambda, q, \lambda) = \Lambda + \frac{1}{2}p^2 - \varepsilon^2 \cos q - \varepsilon^2\beta q \cos \lambda$$

with a trivial scaling of the action variables p, Λ and of the time. In order to remove the unpleasant nonperiodic dependence on the angle q we change the variables via the canonical transformation

$$\Lambda = \Lambda' + \varepsilon^2\beta q' \cos \lambda', \quad \lambda = \lambda',$$

$$p = p' + \varepsilon^2\beta \sin \lambda', \quad q = q',$$

so that, omitting primes and forgetting an unessential constant, the transformed Hamiltonian reads

$$H = \Lambda + \frac{1}{2}p^2 - \varepsilon^2 \cos q + \varepsilon^2\beta p \sin \lambda - \frac{1}{4}\varepsilon^4\beta^2 \cos(2\lambda).$$

This is the Hamiltonian of a perturbed pendulum. We now apply the procedure of Section 2.1. Assuming the condition $\beta < 1/\varepsilon^2$, we try to remove all λ dependent terms via a canonical transformation with generating function

$$\chi = \varepsilon^2\beta p \cos \lambda + \frac{1}{8}\varepsilon^4\beta^2 \sin(2\lambda).$$

The transformation is

$$\Lambda = \Lambda' - \varepsilon^2\beta p' \sin \lambda' + \frac{1}{4}\varepsilon^4\beta \cos(2\lambda'), \quad \lambda = \lambda',$$

$$q = q' - \varepsilon^2\beta \cos \lambda', \quad p = p'.$$

The transformed Hamiltonian has now an infinite expansion; so, still omitting primes and forgetting an

inessential constant, we write only the leading terms. We get the required form (2) with

$$\begin{aligned} h_0 &= \Lambda + \frac{1}{2}p^2, \\ h_1 &= -\varepsilon^2 \cos q + \frac{1}{4}\varepsilon^6 \beta^2 \cos q, \\ h_2 &= -\frac{1}{2}\varepsilon^4 \beta [\sin(q + \lambda) + \sin(q - \lambda)] \\ &\quad + \frac{1}{8}\varepsilon^6 \beta^2 [\cos(q + 2\lambda) + \cos(q - 2\lambda)] \\ &\quad + O(\varepsilon^8 \beta^3); \end{aligned}$$

more precisely, the rest of the expansion has coefficients of the form $\varepsilon^2(\varepsilon^2\beta)^s$. Thus, for $\beta < 1/\varepsilon^2$ this Hamiltonian satisfies condition (iii) of Section 2.1 with $\varepsilon_{\text{eff}} = \varepsilon^2$. With the same consideration of the previous example we conclude that the size of h_{res} is $O(\exp(-1/\varepsilon))$ and that of h_{nonres} is $O(\exp(-\alpha/\varepsilon^2))$, i.e., smaller than h_{res} .

2.3. The quasi-periodically forced pendulum

The Hamiltonian has the form

$$H(I, \varphi, p, q) = \omega \cdot I + \frac{1}{2}p^2 - \varepsilon \cos q + \varepsilon f(\varphi, q) \tag{7}$$

with $(I, \varphi) \in \mathbb{R}^n \times \mathbb{T}^n$, $(p, q) \in \mathbb{R} \times \mathbb{T}$ and ω a diophantine n -vector. We consider the domain \mathcal{G} defined by $|p| < O(\sqrt{\varepsilon})$, so that q is a slow angle, and the resonance module \mathcal{M} has basis $(1, 0, \dots, 0)$. Due to our special choice, this Hamiltonian has already the form 2, with $\varepsilon_{\text{eff}} = \varepsilon$. We now define K looking at the lowest order resonance not belonging to \mathcal{M} . To this end, write the resonance relation as $|lp + k \cdot \omega| < \alpha$. It is evident that the lowest order resonance will be given by $l = 1$ and $p \sim \sqrt{\varepsilon}$. Recalling that a diophantine frequency ω satisfies $|k \cdot \omega| \geq \gamma/|k|^\tau$ for some positive γ and τ , we readily get the relation

$$\gamma/|k|^\tau - \sqrt{\varepsilon} < \alpha.$$

This leads to the natural choice

$$\alpha \sim \sqrt{\varepsilon}, \quad K \sim 1/\varepsilon^{1/(2\tau)}.$$

As a consequence

$$\begin{aligned} h_{\text{res}} &\sim \exp(-1/\varepsilon^{1/(2\tau)}), \\ h_{\text{nonres}} &\sim \exp(-\alpha/\varepsilon) \sim \exp(-1/\varepsilon^{1/2}) \end{aligned}$$

and, recalling that $\tau > n - 1$, the dominating part is h_{res} , provided $n > 1$. The case $n = 1$ requires the construction of Section 2.2.

The case of the whiskered torus can be reduced to the present one with simple considerations.

2.4. The model of Hénon and Heiles

The Hamiltonian is

$$H(x, y, p_x, p_y) = \frac{1}{2}(x^2 + p_x^2) + \frac{1}{2}(y^2 + p_y^2) + x^2 y - \frac{1}{3}y^3, \tag{8}$$

defined in an open ball around the origin. The cartesian variables x, y, p_x, p_y are suitable for analytical estimates. However, formal considerations are easier if one introduces action-angle variables via the canonical transformation $x = \sqrt{2I_1} \cos \varphi_1$, $p_x = \sqrt{2I_1} \sin \varphi_1$, $y = \sqrt{2I_2} \cos \varphi_2$, $p_y = \sqrt{2I_2} \sin \varphi_2$. This gives the Hamiltonian the form

$$H = I_1 + I_2 + f(I_1, I_2, \varphi_1, \varphi_2),$$

where f is a homogeneous polynomial of degree 3 in $\sqrt{I_1}, \sqrt{I_2}$, and a trigonometric polynomial of degree 3 in the angles.

Since the harmonic frequencies are $(1, 1)$, the resonance module \mathcal{M} is generated by the basis $(1, -1)$. In order to give the Hamiltonian the form (2) we perform a Birkhoff normalization. The explicit computation can be found, e.g., in [6]. The normal form starts with a homogeneous polynomial of degree 4 in $\sqrt{I_1}, \sqrt{I_2}$ and a trigonometric polynomial of degree 4 in the angles. Precisely, the normal form up to order 4 has the form $I_1 + I_2 + a(I_1 + I_2)^2 + I_1 I_2 (b + c \cos(2\varphi_1 - 2\varphi_2))$ with suitable coefficients a, b and c . Via the canonical transformation $I_1 + I_2 = J_1$, $I_2 = J_2$, $\varphi_1 = \psi_1$, $\varphi_2 - \varphi_1 = \psi_2$ the Hamiltonian writes

$$H = J_1 + aJ_1^2 + (J_1 - J_2)J_2(a + b \cos(2\psi_2)). \tag{9}$$

From this, one immediately sees that J_1 is a first integral for the truncated Hamiltonian. Looking at the phase portrait in J_2, φ_2 one sees that there are orbits which cover the whole interval $0 < J_2 < J_1$. Thus, the natural choice of \mathcal{G} is $J_1 = \varrho$ and $J_2 \leq \varrho$ for some

positive ϱ . In this domain the frequencies are given by $\omega_1 = 1 + O(\varrho)$ and $\omega_2 = O(\varrho)$. Furthermore, from (9) we have $\varepsilon_{\text{eff}} = \varrho$.

The lowest order resonance in \mathcal{G} which is not contained in \mathcal{M} is $|\omega_1 + k\omega_2| < \alpha$, with a constant α that can be chosen independent of ϱ ; for instance, put $\alpha = \frac{1}{2}$. This gives $k \sim (1 - \alpha)/\varrho$. We conclude that $h_{\text{res}} = O(\exp(-1/\varrho))$. The size of h_{nonres} can be reduced by performing more normalization steps with increasing values of α , as illustrated in Section 2.1. According to the estimates there, we expect to find $h_{\text{nonres}} = O(\exp(-\alpha/\varrho))$, so that it is at most of the same magnitude of h_{res} .

2.5. Neighbourhood of a diophantine KAM torus

According to Kolmogorov, the Hamiltonian in the neighbourhood of a KAM torus can be given the form

$$H(p, q) = \omega_0 \cdot p + f(p, q), \tag{10}$$

where ω_0 satisfies a diophantine condition, and $f(p, q)$ is at least quadratic in p . We shall actually consider the power series expansion in p of the perturbation, and the relevant parameter is the distance from the invariant torus $p = 0$. By the way, this example essentially covers also the case of an elliptic equilibrium point with diophantine harmonic frequencies.

With these assumptions it is natural to define \mathcal{G} as an open ball of radius ϱ and $\mathcal{M} = \{0\}$. Then, we determine K looking for the lowest order resonance in the domain. We know that in a nonlinear system the frequencies generically depend on the actions. Thus, in a ball of radius ϱ , the frequencies can change by a quantity $O(\varrho)$. On the other hand, if ω_0 is a diophantine vector then $|k \cdot \omega_0| \geq \gamma/|k|^\tau$ for some positive γ and τ . Therefore, the lowest order resonance can be estimated in a straightforward way as follows:

$$\begin{aligned} |k \cdot \omega(p)| &= |k \cdot \omega_0 + k \cdot \delta\omega| \geq ||k \cdot \omega_0| - |k \cdot \delta\omega|| \\ &\geq \frac{\gamma}{|k|^\tau} - |k|\varrho \end{aligned}$$

and the condition

$$\gamma/|k|^\tau - |k|\varrho > \alpha$$

is satisfied by all

$$|k| < K \sim (\gamma/2\varrho)^{1/(\tau+1)},$$

provided $\alpha < \varrho$.

By the considerations of Section 2.1, the size of h_{res} is expected to be of order $O(\exp(-1/\varrho^{1/(\tau+1)}))$, so that $\varepsilon_{\text{eff}} = O(\exp(-1/\varrho^{1/(\tau+1)}))$ and $h_{\text{nonres}} = O(\exp(-\alpha/\varepsilon_{\text{eff}}))$. Therefore, we conclude that the leading part of the remainder is h_{res} .

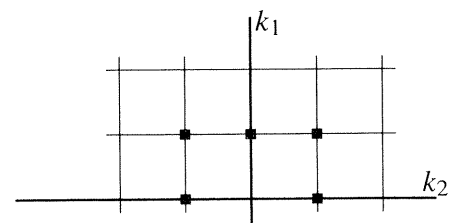
3. Remarks on the “generic” optimality of normal forms

The estimates of the previous section are, in principle, just upper bounds on the coefficients of the Fourier expansion of the Hamiltonian. In this section we give some heuristic arguments, inspired by Chapter V of Poincaré’s *Méthodes nouvelles*, which support the thesis that resonant terms will generically appear, with the size predicted by the estimates above.

For simplicity, let us discuss in detail the non-resonant case $\mathcal{M} = \{0\}$. For instance, consider the Hamiltonian

$$\begin{aligned} H(p, q) &= h(p) + \varepsilon f(q), \\ h(p) &= \omega_1 p_1 + \omega_2 p_2 + \frac{1}{2}(p_1^2 + p_2^2), \\ f(q) &= \cos q_1 + \cos(q_1 - q_2) \\ &\quad + \cos(q_1 + q_2) + \cos q_2. \end{aligned} \tag{11}$$

Let us represent the Fourier modes which appear in f by the diagram



where $k = (k_1, k_2)$ is the Fourier mode, and the dot means that the coefficient of that mode is not zero. Notice that we can always assume $k_1 \geq 0$.

Now we proceed with the first Birkhoff’s normalization step, using the algorithm of Lie series. To this end, we must determine the generating function χ_1 by solving the equation $\{h, \chi_1\} + f = 0$. As is well

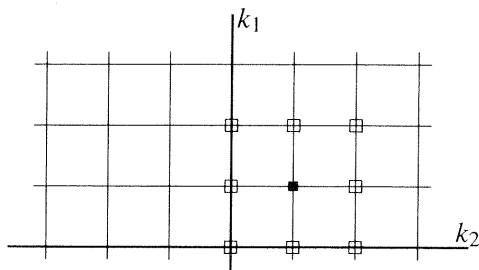
known, χ_1 contains the same Fourier modes as f , and so it is represented by the same diagram as f . Now we determine the transformed Hamiltonian, say,

$$H^{(1)} = h^{(1)} + \varepsilon^2 f_2^{(1)} + \varepsilon^3 f_3^{(1)} + \dots$$

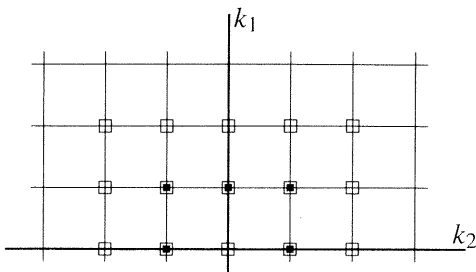
by computing

$$\begin{aligned} H^{(1)} &= \exp(L_{\varepsilon\chi_1})H \\ &= H + \{\varepsilon\chi_1, H\} \\ &\quad + \frac{1}{2}\varepsilon^2\{\chi_1, \{\chi_1, H\}\} + \dots; \end{aligned}$$

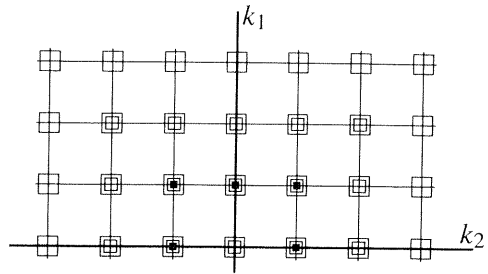
notice that $H^{(1)}$ is already expanded in series of ε . In order to investigate which Fourier modes will appear in $f_2^{(1)}, f_3^{(1)}, \dots$ it is enough to remark that, in a Poisson bracket between two functions, the Fourier modes add algebraically together in pairs. Thus, for instance, the mode (1, 1) in f combined with each mode in χ_1 will produce all coefficients represented by boxes in the diagram



The complete result of $\{\chi_1, f\}$ is represented by the boxes in the diagram



which represents the transformed function $f_2^{(1)}$. Similarly, $f_3^{(1)} = \{\chi_1, f_2^{(1)}\}$ is represented by the diagram



and so on for the next orders. Concerning the size of the coefficients, it is immediate to remark that, unless it happens that some terms cancel out algebraically, $f_2^{(1)}, f_3^{(1)}, \dots, f_s^{(1)}, \dots$ are multiplied by $\varepsilon^2, \varepsilon^3, \dots, \varepsilon^s, \dots$, and contain Fourier modes of order at most $4, 6, \dots, 2s, \dots$ (here, the order of the mode k is $|k| = |k_1| + |k_2|$). Thus, the coefficients decay according to the Fourier estimate $\exp(-|k|\sigma)$ with $\sigma \sim |\ln \varepsilon|$. It is an easy matter to check that the same propagation mechanism works at every step of Birkhoff's normalization procedure.

This shows that all terms of high order are actually generated. In particular, we can expect that *resonant* terms are generated, with coefficient as big as expected according to Fourier's exponential decay.

Being unable to support this heuristic argument with rigorous proofs, we look for numerical evidence. With the aid of a suitably designed algebraic manipulator we perform a number of Birkhoff's normalization steps on a Hamiltonian similar to (11), precisely

$$\begin{aligned} H(p, q) &= \omega_1 p_1 + \omega_2 p_2 + \frac{1}{2}(p_1^2 + p_2^2) \\ &\quad + \frac{1}{10} p_1 p_2 \cos q_1 + \frac{1}{5} p_2^2 \cos(2q_2) \\ &\quad + \frac{2}{5} p_1^2 \cos(q_1 + q_2) \end{aligned}$$

with

$$\omega = (1, \frac{1}{2}(\sqrt{5} - 1)).$$

The computation is performed considering the linear part $\omega_1 p_1 + \omega_2 p_2$ as the unperturbed Hamiltonian, so that we only deal with functions that can be represented as Fourier series in the angles with polynomial coefficients in the actions. All the expansions are truncated at degree 20 in the actions and at $|k| \leq 30$ in the Fourier modes. We proceed by removing angle-dependent terms at increasing orders in p , starting with order 2, and in particular removing only Fourier

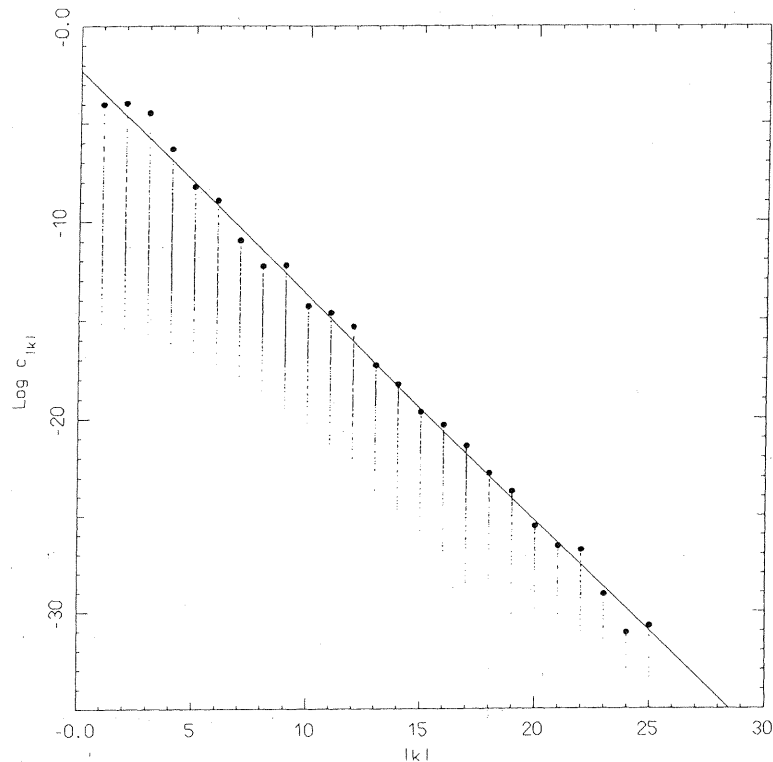


Fig. 1. The size of Fourier coefficients as a function of the Fourier order $|k|$. The label $c_{|k|}$ denotes the norm of $c_{j,k}(p)$ (evaluated over the ball $|p| \leq 0.1$), j being the polynomial order and k the Fourier order. Here we consider only terms corresponding to frequencies $|k_1\omega_1 + k_2\omega_2| < 1$, which are never eliminated by the normal form algorithm. The bold dot marks the maximum of $c_{|k|}$ for each $|k|$. We remark that the distribution of the coefficients follows very well the exponential decay predicted by Fourier law. The line is the least square fit of the bold dots.

modes with $|k \cdot \omega| \geq 0.99$. The latter condition ensures that all modes in the original perturbation are actually removed at first step. Thus, no contribution to h_{res} comes from the original Hamiltonian. Smaller thresholds on $|k \cdot \omega|$ would remove too many terms and would generate terms in h_{res} only at orders incompatible with a practical computation. After some normalization steps we have a series with a general term of the form $c_{j,k} p^j \exp(ik \cdot q)$, namely a monomial in p and a trigonometric monomial in q with a complex coefficient $c_{j,k}$. We plot the quantity $|c_{j,k}| \varrho^{|j|}$ vs. $|k|$, with the particular choice $\varrho = \frac{1}{10}$. This means that we estimate the supremum norm of the coefficient $c_{j,k} p^j$ in the domain $|p| < \varrho$.

Figure 1 shows the result after seven normalization steps; that is, after removing all angle dependent terms with $|k \cdot \omega| > 0.99$ and polynomial degree in p smaller than 10. Actually, performing the next steps does not significantly change the figure. A rough evaluation of the convergence radius in p at this step gives

$\varrho \sim \frac{1}{10}$, which motivates the choice above for the computation of norms. Since we are interested in the structure of h_{res} , in Fig. 1 we plot only the norms of the coefficients of the modes with $|k \cdot \omega| < 0.99$. This makes well evident that resonant modes appear at every order in $|k|$ and that the biggest terms follow very accurately the exponential decay predicted by Fourier's law. In order to underline this fact, we have plotted as a bold dot the maximum of c_k with fixed $|k|$ and computed the least-squares fit of their distribution.

Our computation shows that the scheme of generation of modes illustrated above works effectively, and creates resonant terms according to Fourier's law. It is always possible, of course, that cancellations do occur in particular models (for instance, the Hamiltonian could reveal to be integrable), but in this case a small change in the original Hamiltonian will restore the wanted resonant terms. Let us show how we can easily do it,

Assume that we are considering a given Hamiltonian H in a domain where a given Fourier mode k is resonant, and suppose that, having performed the construction of the normal form, we discover that there is no such mode in the normal form. Let us consider another Hamiltonian $H' = H + a \exp(ik \cdot q)$, with $|a| \sim e^{-|k|\sigma}$; that is, just change the original Hamiltonian by adding the wanted term. Perform on H' the construction of the normal form, up to the same order as for H . We claim that the normal form of H' differs from that of H by exactly that term, plus a quantity smaller by a factor ε .

This is easily seen from the process of construction. Indeed, consider the first normalization step, and denote by χ_1 and χ'_1 the generating functions which put in normal form H and H' , respectively. Since the difference $H' - H$ is a resonant term, it does not contribute to the generating function. Thus, $\chi_1 = \chi'_1$. On the other hand, by linearity, we have $\exp(L_{\varepsilon\chi_1})H' = \exp(L_{\varepsilon\chi_1})H + \exp(L_{\varepsilon\chi_1})a \exp(ik \cdot q)$. Thus, still denoting (with a little abuse) by H and H' the normal form after the first step, the difference is now $H' - H = a \exp(ik \cdot q) + O(\varepsilon e^{-|k|\sigma})$, as claimed. The next normalization steps can introduce only terms smaller than $\varepsilon e^{-|k|\sigma}$, so that they cannot cancel the resonant term. Indeed, in the second step we determine two generating functions χ_2 and χ'_2 . Since the leading part of the difference between the normal forms is the resonant term $a \exp(ik \cdot q)$, which does not contribute to χ'_2 , one has that $\varepsilon^2(\chi'_2 - \chi_2) = O(\varepsilon e^{-|k|\sigma})$. Then, using the trivial identity

$$\begin{aligned} & \exp(L_{\varepsilon^2\chi'_2})H' - \exp(L_{\varepsilon^2\chi_2})H \\ &= \exp(L_{\varepsilon^2\chi_2})(H' - H) \\ & \quad + (\exp(L_{\varepsilon^2\chi'_2}) - \exp(L_{\varepsilon^2\chi_2}))H', \end{aligned}$$

one immediately gets

$$\begin{aligned} & (\exp(L_{\varepsilon^2\chi'_2}) - \exp(L_{\varepsilon^2\chi_2}))H' \\ &= L_{\varepsilon^2(\chi'_2 - \chi_2)}H' + \dots = O(\varepsilon e^{-|k|\sigma}). \end{aligned}$$

On the other hand, $\exp(L_{\varepsilon^2\chi_2})(H' - H) = a \exp(ik \cdot q) + O(\varepsilon^2 e^{-|k|\sigma})$, as claimed above. The same is true also for the subsequent normalization steps.

In order to produce a more formal statement, let us associate to a function $f(p, q) = c_k(p) \exp(ik \cdot q)$ its Fourier norm $\|f\| = \sum_k |c_k|$, where $|c_k|$ is some norm on the coefficient $c_k(p)$ (e.g., the supremum norm over a domain). If f is analytic, then the norm is well defined. Then, by the argument above, we can make the following statement: *for any given Hamiltonian H we can find another Hamiltonian H' , with $\|H' - H\| < e^{-|k|\sigma}$ such that the normal form of H' contains the wanted resonant Fourier mode k with a coefficient of size $e^{-|k|\sigma}$.*

In this sense we speak of “generic optimality of normal forms”.

4. On the size of the chaotic region around a separatrix

A possible application of the results of the previous sections concerns the estimate of the size of the chaotic region around a separatrix of an unperturbed resonance. Indeed, one immediately remarks that there is a striking (and puzzling) agreement between the exponentially small estimates of the remainder of normal forms in Section 2.2 and the estimates of the splitting of separatrices commonly found using the method of Poincaré–Melnikov integral (e.g., in the problem of the rapidly forced pendulum). In this section we show that this phenomenon can be understood on the basis of the remainder of the normal form.

It is quite obvious, in view of the results of Neishtadt [7], that the splitting of separatrices (and therefore also the size of the chaotic region) cannot exceed the size of the remainder of the normal form. Indeed, in all resonant examples in Sections 2.2–2.4 one ends up with an integrable (pendulum like) system with an exponentially small perturbation. Thus, introducing suitable action–angle variables, one proves the existence of invariant tori bounding a strip of exponentially small size around the unperturbed separatrix.

The problem is to find an estimate from below for the splitting, and this is usually done via Melnikov’s theory. However, it is well known that computing Melnikov’s integral for a generic system is a very

difficult problem, perhaps not analytically solvable. We emphasize that in the well-studied case of the rapidly forced pendulum the “optimal” (in the sense of the previous sections) size of the remainder is exactly equal to the size of the splitting rigorously estimated by the method of Poincaré–Melnikov. This suggests that, as a general fact, the size of the chaotic region is the same as the size of the *resonant* remainder, namely h_{res} .

In order to support this point we study the model Hamiltonian with $1\frac{1}{2}$ degree of freedom

$$H(p, q) = \frac{1}{2}p^2 - \varepsilon(1 + \mu \cos(kq - t)) \cos q$$

$$k \sim 1/2\sqrt{\varepsilon}. \quad (12)$$

The choice of this model is dictated by two considerations. Firstly, in view of the estimates on the remainder of the previous sections, the lowest order resonant term is the leading one; so, we discard all nonresonant terms and all resonant terms of larger order. This is reflected in our model, since the unperturbed pendulum $\mu = 0$ contains the low order resonance $p = 0$, and the term $q/(2\sqrt{\varepsilon}) - t$ is the lowest high order resonance encountered in the region around the separatrices of the pendulum. Secondly, any resonant coupling can be typically given a form similar to 12 via elementary trigonometric manipulations.

Avoiding the difficult computation of Melnikov’s integral, we study this model with a method inspired by the theory of adiabatic invariants, as explained in the works of Neishtadt [8], Elskens and Escande [9] and Henrard [10]. The idea is that the pendulum of our model has a pulsating separatrix which sweeps a region of size μ . However, the theory of adiabatic invariants can be applied only if the separatrix is slowly pulsating, as in the model $H = \frac{1}{2}p^2 - (1 + \mu \cos(\varepsilon t)) \cos q$. Conversely, in our case the frequency of pulsation is slow only when p is close to $2\sqrt{\varepsilon}$, and becomes fast when $\dot{q} = p$ decreases to zero. The effect of the fast pulsation could be possibly averaged out, thus giving contributions of higher order in μ . In the following, we show that the time of slow pulsation is long enough to produce effects of adiabatic capture.

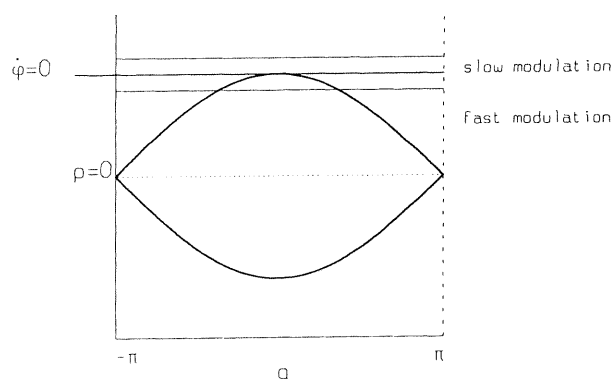


Fig. 2. Schematic representation of the dynamics of Hamiltonian 12. The separatrix of the pendulum can be considered to be slowly pulsating only when p is close to $2\sqrt{\varepsilon}$; otherwise, the pulsation is fast and can be averaged out to higher order in μ . The time spent in the band where the pulsation is small, is long enough to allow capture of particles in a strip $O(\mu)$. See text for the obtention of formulas.

Referring to Fig. 2, we look for the domain in space and time where the time derivative of $\varphi := q/(2\sqrt{\varepsilon}) - t$ is smaller than \dot{q} . The motion of q close to the separatrix apex $p = 2\sqrt{\varepsilon}$, $q = 0$ is approximated by $q(t) = 2\sqrt{\varepsilon}t - O(\varepsilon^{3/2}t^3)$, so that $\dot{q} = 2\sqrt{\varepsilon} - O(\varepsilon^{3/2}t^2)$. Consequently, $\varphi = O(\varepsilon t^3)$, and $\dot{\varphi} = O(\varepsilon t^2)$. Therefore, we have $\dot{\varphi} < \dot{q}$ for all $t < O(1/\varepsilon^{1/4})$. During this time the increment of both φ and q is $O(\varepsilon^{1/4})$, and so the change of the quantity $\mu \cos \varphi$ is at least $O(\mu\varepsilon^{1/2})$. This shows that all phase points closer to the unperturbed separatrix by less than the latter quantity can be captured (expulsed) due to the slow pulsation of the separatrix. We conclude that the size of the chaotic region is estimated to be at least $O(\varepsilon^{1/2}\mu)$ close to the apex of the separatrix, and at least $O(\varepsilon^{1/4}\mu^{1/2})$ close to the unstable equilibria.

For $k > 1/(2\sqrt{\varepsilon})$ a similar argument applies, as the reader can easily check. Conversely, for $k < 1/(2\sqrt{\varepsilon})$ the mode $kq - t$ is no more resonant in the domain characterized by the separatrix. Thus, the argument above fails, and one expects that the effect of the perturbation is averaged out by the fast oscillations. In this case, we can apply normal form theory, and so we expect that the effect of the perturbation is estimated by the size of the first high order resonant term generated during the construction of the normal form. The first high order resonant term is $q/(2\sqrt{\varepsilon}) - t$, and the corresponding resonant term is produced

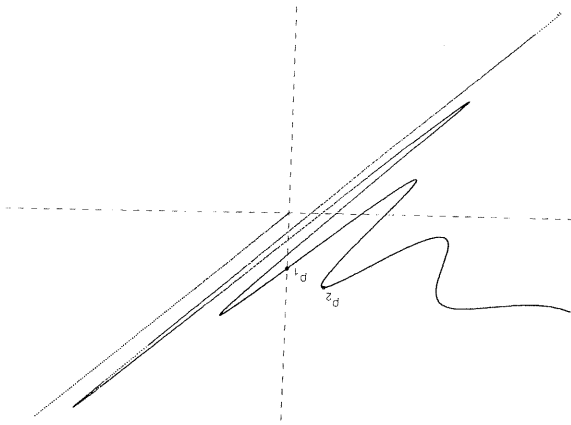


Fig. 3. The determination of the splitting of separatrices: p_1 is the first intersection of the unstable manifold with the vertical axis passing through the unstable equilibrium; p_2 is the maximum of p along the last lobe. We take $dp = \max(p_1, p_2)$ as a measure of the size of the separatrix splitting.

after $1/(2\sqrt{\epsilon}) - k$ steps. According to the estimates of the previous section, its coefficient is expected to be $O(\exp(k - 1/(2\sqrt{\epsilon})))$.

In conclusion, we expect to observe two different regimes: for $k \geq (2\sqrt{\epsilon})$ the splitting is at least $O(\epsilon^{1/2}\mu)$; for $k < (2\sqrt{\epsilon})$ the splitting is at least $O(\exp(k - 1/(2\sqrt{\epsilon})))$. In the latter case, an upper bound of roughly the same size can be predicted according to Neishtadt's estimates [7].

We look now for a numerical confirmation of our estimates. To this end, for given values of ϵ , μ and k , we consider the Poincaré section at $t = 2m\pi$ ($m = 0, 1, \dots$), and locate the unstable fixed point, \bar{q} , \bar{p} say, of that mapping close to the unperturbed fixed point $q = -\pi$, $p = 0$. We construct the unstable manifold originating from \bar{q} , \bar{p} up to its first intersection with the vertical axis $q = \bar{q}$. Referring to Fig. 3, we determine the ordinate p_1 of the intersection, and also the ordinate p_2 of the maximum of the last encountered lobe. Then we use the maximum dp between $p_1 - \bar{p}$ and $p_2 - \bar{p}$ as indicator of the size of the separatrix splitting.

The results for $\epsilon = 10^{-4}$, $k = 50$ and μ in the range $[10^{-12}, 10^{-3}]$ are reported in Fig. 4, in log-log scale. As one sees, the points are well aligned, and the slope of the straight line computed by least-squares fit is 0.4989628. This is in excellent agreement with

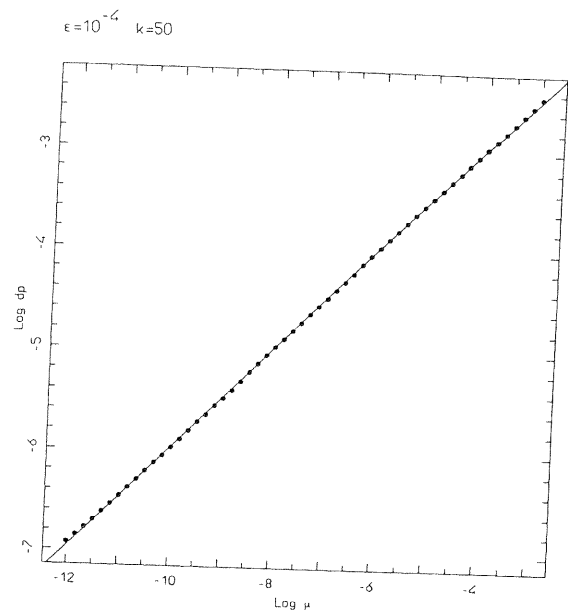


Fig. 4. The size of the separatrix splitting for Hamiltonian (12) as a function of μ . ϵ is fixed, equal to 10^{-4} and k is fixed equal to $2\sqrt{\epsilon}$. We remark the power decay of dp as a function of μ . The line is the least-squares fit in log-log scale, and its slope is very close to $\frac{1}{2}$, which is the value predicted by the theory.

our predicted scaling $\sqrt{\mu}$, thus fully confirming our estimate from below.

Concerning the dependence on k , we show in Fig. 5 the results for $\epsilon = \mu = 10^{-4}$. It is immediately seen that there is a maximum close to $k = 50$, with a sharp asymptotically exponential decay for decreasing k . Conversely, all $k > 50$ present a similar splitting, in agreement with the prediction above. For values of k less than 30 numerical errors appear to dominate.

5. Conclusions and discussion

We have pointed out that the estimates for the remainder of normal forms can be easily understood and predicted exploiting the fact that exponentially small high order resonances are always present in any domain, and that these resonant terms cannot be eliminated, and dominate all the rest. This picture appears to be generically optimal, in the sense that any Hamiltonian can be forced to produce high order resonant terms, eventually adding small corrections.

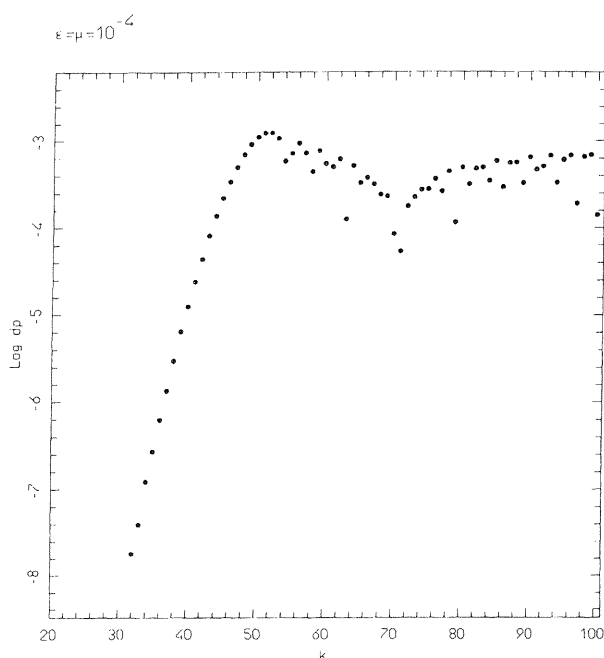


Fig. 5. The size of the separatrix splitting for Hamiltonian (12) as a function of k . The parameters ε and μ are fixed. We remark the sharp decay when $k < 50$; Conversely the size of the splitting is constant (within one order of magnitude) when $k \geq 50$.

As a natural application we have discussed some problems related to separatrix splitting. The main argument is that high order resonant terms are effective in producing a splitting which is proportional to their actual size. Thus, the size of splitting must be at least proportional to the size of the remainder h_{res} of the normal form. Since this remainder does generically exist (in the sense of Section 3), we expect that the splitting does generically occur, with an exponentially small size. We believe that our argument can be made rigorous, so as to actually produce an estimate from below of the splitting in the generic case.

We discuss now the relation of our results with the known method of Melnikov's integral. Firstly, it is evident that Melnikov's method, being based on an explicit evaluation of relevant quantities, can produce a definite result on a given problem; in contrast, our method leads only to conclusions which generically apply to a wide class of problems, but turns out to be unable to give a precise information on a given specific model. On the other hand we believe that our approach is interesting in the framework of Hamiltonian dynamical systems, inasmuch one is interested in a generic

behaviour. Secondly, the normal form method brings some light on some hidden mechanisms of Melnikov's method. Indeed, one is confronted with the mysterious fact that big perturbations produce only exponentially small effects. In the light of normal form theory, we know that most of the effect of the perturbation is a quasiperiodic oscillation which can be removed by introducing suitable coordinates; according to the normal form above, chaotic effects are due to high order resonances, so that they are exponentially small. This fits perfectly with the remark, made a long time ago by Contopoulos [11] and Chirikov [12], that chaotic behaviour arises in connection with interaction of resonances. In this light, the method of Melnikov seems to succeed in averaging out all quasiperiodic behaviour. In this connection, it could be interesting to use the method of Melnikov's integral in order to compute the splitting of separatrices for the Hamiltonian (12).

We believe that our approach can also provide interesting indications on two problems which are still open in the framework of Melnikov's method.

The first problem concerns the computation of the splitting when the perturbation is an infinite series. The problem arises by the fact that the usual method of evaluation of Melnikov's integral by residues seems to fail, due to essential singularities in the series (see for instance [13]). Conversely, the argument discussed in Section 3 on the generic size of the resonant terms in the remainder of normal forms, is based only on the analyticity properties of the original Hamiltonian, so that it holds also for infinite series. As well, the property that the size of the chaotic region is the same as the size of the resonant remainder (Section 4), is independent of the number of terms actually present (the remainder is usually made of an infinite analytic series). So, we conclude that even in the case where the original Hamiltonian is an infinite series one has to expect in general an exponentially small splitting as in the truncated case.

The second problem concerns the special case of the rapidly forced pendulum (6) in the case $\beta = \varepsilon^s$. Big effort has been made in order to investigate the range of s for which there is an exponentially small splitting. For instance, Holmes et al. [13] showed the result for $s \geq 8$, and more recently the condition was improved

to $s \geq 0$ by Delshams and Seara [16]. Furthermore, numerical computations by Benseny and Olivé [17] show that one should be able to improve up to $s \geq -1$. The question is whether one can improve up to $s > -2$. Our consideration in Section 2.2 show that the splitting is indeed expected to be $O(\exp(-1/\varepsilon))$ for $s > -2$ for the specific model (6); conversely, different forms of the coupling term can produce different powers of ε inside the exponential formula for different values of s ; for instance, the model (4) gives a splitting $O(\exp(-1/\varepsilon^{1/2}))$ for $s \geq -\frac{1}{2}$ and $O(\exp(-1/\varepsilon^{1+s}))$ for $-1 < s < -\frac{1}{2}$.

Acknowledgements

We wish to thank Prof. A. Delsham for the useful discussions that we had on separatrix splitting matters. This work has been supported by E.C. contracts N. CHRX-CT93-0330/DC and N. ER-BCHRXCT940460.

References

- [1] N.N. Nekhoroshev, Exponential estimates of the stability time of near-integrable Hamiltonian systems, *Russian Math. Surveys* 32 (1977) 1.
- [2] N.N. Nekhoroshev, Exponential estimates of the stability time of near-integrable Hamiltonian systems, 2, *Trudy Sem. Petrovsk.* 5 (1979) 5.
- [3] G. Benettin, L. Galgani and A. Giorgilli, A proof of Nekhoroshev's theorem for the stability times in nearly integrable Hamiltonian systems, *Celestial Mech.* 37 (1985) 1–25.
- [4] A. Giorgilli and E. Zehnder, Exponential stability for time dependent potentials, *ZAMP* 43 (1992) 827–855.
- [5] A.N. Kolmogorov, On the preservation of conditionally periodic motions, *Dokl. Akad. Nauk SSSR* 98 (1954) 527.
- [6] F.G. Gustavson, On constructing formal integrals of a Hamiltonian system near an equilibrium point, *Astronom. J.* 71 (1966) 670–686.
- [7] A.I. Neishtadt, The separation of motions in systems with rapidly rotating phase, *Prikl. Matem. Mekh.* 48 (2) (1984) 197–204; English transl.: *J. Appl. Math. Mech.* 48 (2) (1984) 133–139.
- [8] A.I. Neishtadt, Passage through a separatrix in a resonance problem with a slowly varying parameter, *Prikl. Mat. Mekh.* 39 (1975) 621–632.
- [9] Y. Elskens and D.F. Escande, Slowly pulsating separatrices sweep homoclinic tangles where islands must be small: An extension of classical adiabatic theory, *Nonlinearity* 4 (1991) 615–667.
- [10] J. Henrard, The adiabatic invariant in classical mechanics, *Dynamics Reported* 2 (1993) 117–235.
- [11] G. Contopoulos, in: *Les nouvelles méthodes de la dynamique stellaire*, eds. F. Nahon and M. Hénon, CNRS, Paris (1966); see also *Bull. Astronom. Ser.* 3, 2 Fasc. 1 (1967) 233.
- [12] B.V. Chirikov, A universal instability of many dimensional oscillator system, *Phys. Rep.* 52 (1979) 263–379.
- [13] P. Holmes, J. Marsden and J. Scheurle, Exponentially small splitting of separatrices with applications to KAM theory and degenerate bifurcations, *Contemp. Math.* 81 (1988) 213–244.
- [14] V.G. Gelfreich, Separatrices splitting for the rapidly forced pendulum, in: *Proc. Dynamical Systems Semester*, eds. S. Kuksin, V.F. Lazutkin and J. Pöschel, St. Petersburg, 1991 (Birkhäuser, Basel, 1993).
- [15] J.A. Ellison, M. Kummer and A.W. Sáenz, Transcendentally small transversality in the rapidly forced pendulum, *J. Dynamics and Differential Equation* 5 (2) (1993) 241–277.
- [16] A. Delshams and T.M. Seara, An asymptotic expression for the splitting of separatrices of the rapidly forced pendulum, *Commun. Math. Phys.* 150 (1992) 433–463.
- [17] A. Benseny and A. Olivé, High precision angles between invariant manifolds for rapidly forced Hamiltonian systems, in: *Int. Conf. on Differential Equations*, eds. C. Perelló, C. Simó and Solá-Morales, Barcelona, 1991 (World Scientific, Singapore, 1993) pp. 315–319.